IS THERE A GRAVITATIONAL COLLAPSE OF THE WAVE–PACKET ?

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1 Introduction

In this paper, we discuss a possible solution of the measurement problem, which might be called the "Penrose scenario", which combines three ideas:

- Modifying ordinary non-relativistic time evolution by adding non–linear terms due to self-gravity of particles[6],[1],
- Showing that macroscopic states of minimal gravitational energy are well–localized in phase–space[1],
- Explaining the decay into these localized states by emission of gravitational radiation.

The last idea has, to our knowledge, not yet been discussed in the literature. Penrose[6] considered a decay–time of the order of magnitude $\hbar/E_{grav}$ without being explicit about the physical nature of the decay. Diósi[2] considered continuous measurements which lead to stochastic evolution equations but didn't provide a concrete mechanism for these measurement processes. Since the emission of gravitational radiation due to oscillating quadrupole moments of mass density is – according to General Relativity – a well-understood and inevitable process, we don't think, that we have added some strange element to the first two ideas, and feel entitled to still call the whole approach the "Penrose scenario". The problem, of course, is whether this effect could be of any relevance.

2 Gravitational self–interaction (GSI)

Usually, in non–relativistic quantum theory the gravitational field is, like the electro–magnetic field, treated as semi–quantized: The external gravitational field is approximated by a classical field, but the interaction between two particles is described by a potential energy

$$V(\mathbf{r}_1, \mathbf{r}_2) = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad . \tag{1}$$

This potential energy doesn't have an objective value in most situations, it is an observable with an expectation value $\langle \psi | V \psi \rangle$ where $\psi(\mathbf{r}_1, \mathbf{r}_2)$ is the two–particle wave–function.

Since gravity is not just another interaction but represents the influence of curved space–time, it seems to be legitimate to consider alternatives to the semi–quantized approach. In this paper, we will adopt the extreme alternative to consider the gravitational field as completely classical, not only with respect to external fields but also for the interaction between particles. This means that the gravitational interaction potential appearing in Schrödinger's N-particle equation is chosen as [a]

$$\frac{1}{2} \sum_{i=1}^{N} m_i V(\mathbf{r}_i) \quad , \tag{2}$$

where $V(\mathbf{r})$ is the solution of the classical field equation (with suitable boundary conditions)

$$\Delta V = -4\pi G \rho \quad , \tag{3}$$

and the mass density ρ is given by

$$\rho(\mathbf{r}, t) = \sum_{j=1}^{N} m_j \int |\psi(\mathbf{r}_1, \ldots, \mathbf{r}, \ldots, \mathbf{r}_N, t)|^2 d\mathbf{r}_1 \ldots \widehat{d\mathbf{r}_j} \ldots d\mathbf{r}_N \quad . \tag{4}$$

This means that we consider some kind of Hartree–Fock approximation to be the exact equation, but, different from the usual Hartree–Fock ansatz, also self–interaction occurs. Thus, even for a one–particle problem in empty space, a non–linear extra term will give rise to some attractive force, which will eventually stop the dispersion of the wave packet.

2.1 GSI for single particles

The eigenvalue problem of the modified non–linear Schrödinger equation has been treated numerically by Moroz et al.[5]. The existence and uniqueness of

[a] The factor $\frac{1}{2}$ is introduced to obtain the correct classical limit of the gravitational interaction of N particles.

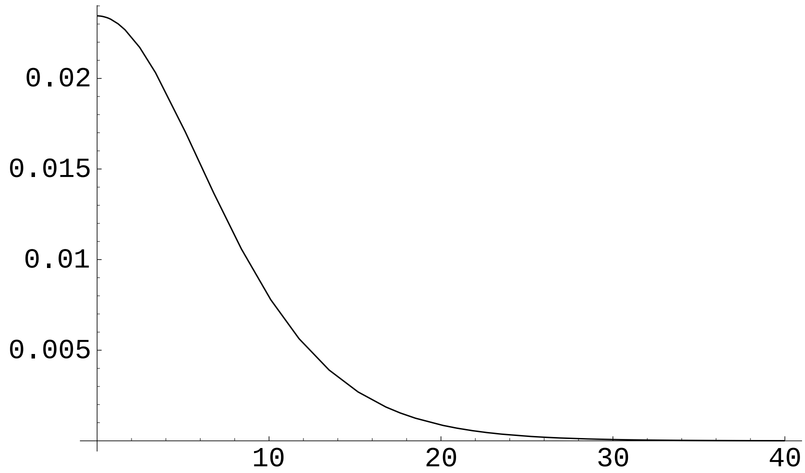

Figure 1. Ground state $\psi_0(r/R_d)$ of a single particle with GSI.

the ground state has been shown by E. Lieb[3] and P. Tod/I. M. Moroz[7] using different methods. Since the modified Schrödinger equation is invariant under Galileo transformations, the ground state solution will represent a localized particle travelling through space with constant velocity (a "soliton") described in its rest frame. Superpositions of different ground states (i. e. translated and/or boosted ones) will not be stationary states nor states of minimal energy because of the non–linearity of the Hamiltonian. These, like the higher eigenstates, would not be stable but decay into a state of minimal energy by emitting gravitational radiation. Could these effects be observed? It follows already from dimensional analysis that the spatial extension of the ground state will be of order [b]

$$R_d \stackrel{\text{def}}{=} \frac{\hbar^2}{Gm^3} \quad , \tag{5}$$

and the binding energy will be of order

$$E_d \stackrel{\text{def}}{=} \frac{G^2 m^5}{\hbar^2} \quad . \tag{6}$$

[b] The "d" stands for "dwarf", since the balance between Schrödinger pressure and gravitational attraction in the ground state is analogous to the stability of a white dwarf in astrophysics.

The more precise values are [c]

$$\sqrt{\langle r^2 \rangle} = 9.3 R_d \quad , \tag{7}$$

and for the energy eigenvalue:

$$E = (\Leftrightarrow 0.0375 \pm 0.001) E_d \quad . \tag{8}$$

Hence, for elementary particles and atoms R_d will be of cosmological dimension and $E_d \approx 10^{-67}$ eV and the non–linear terms can be neglected for any realistic application.

2.2 GSI for macroscopic systems

For macroscopic masses, R_d would be considerably small (a Planck mass of $2 \cdot 10^{-8}$ kg would have an R_d of Planck length, $2 \cdot 10^{-35}$ m), but objects of such mass are usually many–particle systems and cannot be described by a one–particle equation.

However, under certain assumptions, it can be shown heuristically that a macroscopic object will be localized too (in the same sense as above, that superpositions of different localized states will be unstable). We consider a system of N identical nuclei of mass Am_p (A: atomic weight, m_p: proton/neutron mass) and K electrons with total mass

$$M \stackrel{\text{def}}{=} \sum_{i=1}^{N+K} m_i \approx NAm_p \quad . \tag{9}$$

We will suppress spin and tacitly assume (anti–)symmetrizing of the $N+K$–particle wave–function $\psi(\mathbf{r}_1, \ldots \mathbf{r}_{N+K})$. Let

$$\mathbf{R} \stackrel{\text{def}}{=} \frac{1}{M} \sum_{i=1}^{N+K} m_i \mathbf{r}_i \approx \frac{1}{N} \sum_{i=1}^{N} \mathbf{r}_i \tag{10}$$

be centre of mass coordinates and

$$\mathbf{s}_i \stackrel{\text{def}}{=} \mathbf{r}_{i+1} \Leftrightarrow \mathbf{r}_i \qquad i = 2 \ldots N+K \tag{11}$$

the relative coordinates. We are looking for eigenstates of the corresponding (modified) Hamilton operator which are of the product form

$$\psi(\mathbf{r}_1, \ldots \mathbf{r}_{N+K}) = \Psi(\mathbf{R}) \varphi(\mathbf{s}) \quad , \tag{12}$$

[c] Carsten Hartmann, private communication. Taking into account the factor 1/2 we have introduced earlier, this is compatible with the findings in Moroz et al. [5].

where $\varphi(\mathbf{s})$ is the ground state of the $\mathbf{s}$– part of the Hamiltonian $H_{\rm rel}$ and $\Psi(\mathbf{R})$ is spherical symmetric and concentrated essentially within a sphere of radius R_0. According to equation (2) and (3), the non–linear term in the Hamiltonian reads

$$\tilde{V} = -G \sum_{i,j=1}^{N} m_i m_j \int \frac{|\Psi(\mathbf{R}')|^2 |\tilde{\varphi}(\mathbf{s}')|^2}{|\mathbf{r}_i(\mathbf{R}, \tilde{\mathbf{s}}) - \mathbf{r}_j(\mathbf{R}', \tilde{\mathbf{s}}')|} d\mathbf{R}' d\tilde{\mathbf{s}}' \ , \qquad (13)$$

where we have neglected the terms involving electrons and consequently denoted by $\tilde{\varphi}(\tilde{\mathbf{s}})$ the part involving only relative coordinates of nuclei. Now we come to a crucial hypothesis, called (P):

> We shall assume that $\tilde{\varphi}(\tilde{\mathbf{s}})$ is sharply peaked about some lattice configuration $\hat{\mathbf{s}}$ with a mean distance $\delta \approx A^{\frac{1}{5}} 10^{-10}$m between neighbouring nuclei.[9]

If R_0, which measures the spatial extension of $\Psi(\mathbf{R})$, is smaller than δ, the smeared–out lattice sites will practically not overlap and the terms of (13) with $i \neq j$ do not depend on $\Psi(\mathbf{R})$ but only on $\hat{\mathbf{s}}$. These terms could be included in $H_{\rm rel}$ and neglected, because they are smaller than the Coulomb terms by a factor of 10^{-41}. For the terms with $i = j$ we may use the approximation

$$|\mathbf{r}_i(\mathbf{R}, \tilde{\mathbf{s}}) - \mathbf{r}_i(\mathbf{R}', \tilde{\mathbf{s}}')| \approx |\mathbf{r}_i(\mathbf{R}, \hat{\mathbf{s}}) - \mathbf{r}_i(\mathbf{R}', \hat{\mathbf{s}})| = |\mathbf{R} - \mathbf{R}'| \ , \qquad (14)$$

which yields

$$\tilde{V} \approx -GN(M/N)^2 \int \frac{|\Psi(\mathbf{R}')|^2}{|\mathbf{R} - \mathbf{R}'|} d\mathbf{R}' \ . \qquad (15)$$

Hence $\Psi(\mathbf{R})$ approximately satisfies the eigenvalue equation of a one–particle Hamiltonian with mass M and reduced gravitational constant $G' = G/N$. We may employ our previous result on the ground state of that problem and obtain

$$R_0 \approx 10 \frac{\hbar^2}{G(m_p A)^3} \frac{1}{N^2} \ . \qquad (16)$$

We define a critical number of nuclei N_c by $N > N_c \Leftrightarrow R_0 < \delta$. For $N < N_c$ localization still occurs, but our simplified approach no longer applies. The following table shows some typical values for N_c, the total mass M_c and the binding energy E:

In this analysis we have neglected other macroscopic degrees of freedom besides $\mathbf{R}$, like the orientation of the whole system, but we think that a more

Table 1.

Element	A	N_c	M_c/kg	E/J
C	12	10^{15}	$2 \cdot 10^{-11}$	$1 \cdot 10^{-36}$
Fe	56	10^{14}	$9 \cdot 10^{-12}$	$2 \cdot 10^{-36}$
Au	197	10^{13}	$4 \cdot 10^{-12}$	$4 \cdot 10^{-36}$

refined account would not change the estimates of the order of magnitudes we have obtained.

3 Estimate of the decay–time

As mentioned above, superpositions of localized states will not be stationary. We expect that they will oscillate around the localized states with a relative amplitude of ~ 1 and a frequency $\omega = E/\hbar$, where E is of the order of magnitude of the gravitational energy. This can be confirmed by numerical studies, and, to some extent, by perturbation theory. We thus may set

$$E = (\frac{G}{N})^2 \frac{M^5}{\hbar^2} = \hbar\omega \quad , \tag{17}$$

and

$$T = \frac{E}{P} \quad , \tag{18}$$

where T is the decay time and P the mean power of gravitational radiation. Neglecting numerical constants, the latter is given by (see, for example, Wald[8](4.4.60))

$$P = \frac{G}{c^5} M^2 L^4 \omega^6 \quad , \tag{19}$$

where L denotes the spatial separation of the superposed states. We may set

$$M = N m_{\text{nucl}} \quad , \tag{20}$$

and obtain, using PLANCK units T_p, L_p, M_p

$$\frac{T}{T_p}(\frac{L}{L_p})^4 = N^{-17}(\frac{M_p}{m_{\text{nucl}}})^{27} \quad . \tag{21}$$

To give a numerical example, we choose some small, but macroscopic values of $T = 10^{-3}s$, $L = 10^{-4}m$ and nucl= Fe^{56} and obtain a small, but still macroscopic mass of $M = 4M_p$ and to $N = 10^{18}$ nuclei. The corresponding energy is $E = 10^{-5}eV$ and the wave-length of the gravitational radiation

($\lambda \approx 10^{-2}m$) is still large compared with L, as it must be for the slow motion approximation used to derive (19).
Note that the large power N^{-17} could easily absorb other reasonable values of L and T. Of course, these crude estimates are not enough to prove that the "gravitational collapse of the wave packet" could really occur. For this more detailed studies would be necessary. But they show that the Penrose scenario, as it is presented here, cannot be ruled out from the outset.
At this point we may speculate about a new distinction between the microscopic and the macroscopic realm: For microscopic systems alone, localization and decay is negligible. For large macroscopic systems, the decay into localized states is immediate, thus the superposition principle can be neglected. For certain interactions between microsystems and macrosystems, called measurement processes, localization requires some small macroscopic time.

4 Open Problems

The model of the measurement process sketched here is a deterministic one, as long as gravitational fields are taken as purely classical. What then is the origin of randomness which is so characteristic for quantum measurements? At the moment one could only speculate about this. It could be that by amplification, which is a typical feature for measurement processes, a larger and larger part of the environment is brought into a state of oscillating superposition. Due to the large power N^{-17} in (21), the gravitational decay is, so to speak, switched on at some instant of time and a flash of gravitational radiation is emitted. The localized states are local minima of gravitational energy in the unit sphere of HILBERT–space and are surrounded by basins of attraction corresponding to dissipation of enery by radiation. It could be, that the result of the measurement depends on which basin is occupied by the system in the very moment when gravitational effects become strong.
At any case, it should be clear, that a complete theory of the measurement following these lines has to explain, why certain superpositions give results with probabilities according to the rules of ordinary quantum theory. BORN's statistical interpretation could no longer be imposed but should be proved to hold, at least in some limiting sense.
A further objective is to write down a modified SCHRÖDINGER equation which not only accounts for non–linear effects of self–gravity but also for dissipative effects due to the emission of gravitational radiation discussed above. This could be done in a heuristical manner by adding a term which gives rise to energy losses compatible with (19). A more satisfactory approach would be to derive some post–Newtonian approximation to SCHRÖDINGER's equation

allowing for these effects.
Only if some of these problems are solved, one could proceed to consider simplified simulations of concrete measurements like Stern-Gerlach measurement of spin, EPR-situations or pairs of particle detectors. In order to reproduce the quantum mechanical correlations occuring in such experiments, the modified Schrödinger equation would have to show some non-local features. Recent work on the connection between non-linearity and non-locality[4] gives rise to some optimism in this respect.

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