Nonlinear wave equations
(uncomplete draft)

Heinz-Jürgen Schmidt

March 12, 2003

Contents

1 Introduction 2

2 Linear equations 4
  2.1 Mathematical definitions ........................................... 4
  2.2 Physical examples and interpretation ................................... 5
    2.2.1 Harmonical oscillator ............................................. 5
    2.2.2 Electromagnetic waves in a linear medium ......................... 7
    2.2.3 Schrödinger equation .............................................. 7
    2.2.4 Linear dispersion .................................................. 7

3 Nonlinear equations 7
  3.1 Generalities ...................................................... 7
  3.2 Nonlinear approximation to linear equations .......................... 7
  3.3 Nonlinear 1-dimensional oscillations .................................. 7
    3.3.1 Small damping, no driving force .................................. 7
    3.3.2 General case: chaotic motion ..................................... 8
    3.3.3 The Duffing oscillator (1918) ..................................... 8

4 Cubic Schrödinger equation (CSE) 10
  4.1 Derivation from a Duffing-Lorentz model ................................ 10
  4.2 General properties .................................................... 14
  4.3 Conservation laws ...................................................... 15
  4.4 1-soliton solutions ..................................................... 17
  4.5 Interaction of 2 solitons ............................................... 18

5 Korteweg-de Vries equation 19
  5.1 1-soliton solutions .................................................... 19
  5.2 A physical derivation of the KdV equation for water waves ........ 21
1 Introduction

This course addresses itself mainly to the fellows of the graduate college “Nonlinearities of optical materials”. It aims at providing a general physical background rather than dealing with the details of the various projects of this graduate college.

It is generally held that most phenomena of nature are, in principle, nonlinear. Linearity is considered to occur only under idealized special circumstances. For example, any real oscillation will be nonlinear, and can only be approximated by a linear (or harmonic) oscillation in the case of small amplitudes. In contrast to this general situation, most methods of theoretical physics are confined to the linear case, which is mathematically much easier to handle than the nonlinear case. Yet the abundance of nonlinearity seems somewhat paradoxical in view of the fundamental theories of physics which are, at present, general relativity and quantum field theory, including gauge theories of the fundamental interactions. These are indeed nonlinear theories, but their limiting cases which are believed to describe condensed matter and electromagnetic waves, are linear. Among the limiting cases we have the quantum theory of atoms, molecules and solid states as well as electrodynamics, quantum or classical. What then is the physical origin of nonlinearity? We will frequently come back to this question during this course.

The first part of the course will deal with the definition of linearity and nonlinearity of (wave) equations and a couple of physical examples. The definition is not as trivial as it may look at first sight, since a nonlinear transformation can turn a linear equation into a nonlinear one. One wouldn’t like to call the result a nonlinear problem, since it is only a linear equation in some nonlinear disguise. But this leads to very difficult problems. In this part of the course we will consider 3 typical examples of linear equations of motion or wave equations and recall some approaches and methods of solution which are typical for linear equations. Then we will consider 3 nonlinear counter-parts of these equations where the mentioned methods fail.
One of the central themes of this course is the question: What are the physical origins of nonlinearity? We will have to give different answers depending on the case under consideration. From the huge class of nonlinear wave equations in physics we will select two important cases: The Korteweg-de Vries (KdV) equation

\[ \psi_t + \sigma \psi \psi_x + \psi_{xxx} = 0, \]  

(1)

and the cubic Schrödinger (CSE) equation

\[ i\psi_t + \psi_{xx} + \sigma |\psi|^2 \psi = 0. \]  

(2)

Here \( \sigma \) denotes a constant and the subscripts denote partial derivatives with respect to \( x \) or \( t \). Both equations have numerous physical applications. We will concentrate on two typical applications and present detailed derivations from more fundamental equations in both cases. Thereby we study the physical origins of nonlinearity and, moreover, techniques of approximation.

The KdV equation is suited to describe waves in shallow water. Among its solutions are the solitary waves (or “solitons”) first reported by J. Scott Russell in 1843. The nonlinearity of the KdV equation can be traced back to the nonlinearity of the more fundamental Navier-Stokes equations. The CSE appears, for example, in the context of self-focussing of laser beams in nonlinear optical media. It describes the first nonlinear correction to linear polarization and the propagation of certain optical solitons.

As a rule, nonlinear equations cannot be solved exactly. Being one exception to this rule, the exact solutions of some nonlinear wave equations (including the KdV- and the CSE) describing the interaction of \( N \) solitons has been one of the high-lights of mathematical physics of the 20th century. Unfortunately, the pertinent techniques (Lax theory, inverse scattering method, Bäcklund transforms etc.) are too complicated to be presented in detail in this course. Rather we will try to give a rough overview about the methods involved and show some results in graphical form.

For a selected number of topics we provide numerical and analytical studies in the form of MATHEMATICA notebooks, see the Appendices.

**Acknowledgement**

I thank M. Kadiroglu for his aid in preparing this course and F. Homann for providing the material of app. 5 and critical reading of the manuscript.
2 Linear equations

2.1 Mathematical definitions

The definition of the linearity of any (wave) equation which has to comprise numerous physical cases is necessarily very abstract. We will illustrate this abstract definition by a couple of physical examples.

We assume as known the notion of (real or complex) linear space or vector space. Its definition is taught in the introductory courses on linear algebra, see e. g. [1]. We will only consider the complex case, the linear case being analogous.

Let $V, W$ be $\mathbb{C}$-linear spaces and $D : V \rightarrow W$ a linear operator. Then a linear, homogeneous (wave) equation is of the form

$$D \phi = 0, \; \phi \in V.$$  \hfill (3)

One may think of $\phi$ as a field obeying the wave equation. $D$ will also be called the “wave operator”.

It follows that the set of solutions of (3), $Sol_h \subset V$, will be a linear subspace of $V$. This means that $\phi_1, \phi_2 \in Sol_h$ and $c_1, c_2 \in \mathbb{C}$ implies

$$c_1 \phi_1 + c_2 \phi_2 \in Sol_h.$$  \hfill (4)

This equation expresses some kind of superposition principle for fields satisfying a linear, homogeneous wave equation. The corresponding inhomogeneous linear (wave) equation will be of the form

$$D \phi = \rho, \; \phi \in V, \; \rho \in W.$$  \hfill (5)

Mathematically, the inhomogeneous equation can also be written in the form (3), but its physical interpretation is different: $\rho$ is assumed to be given, and $\phi$ is to be determined by $\rho$, (5) and additional boundary conditions. Typically, $\rho$ is the “source” and $\phi$ is the “field” produced by $\rho$. In general, $\phi$ is not uniquely determined by $\rho$ and (5), since $D \phi_1 = \rho = D \phi_2$ implies $D (\phi_1 - \phi_2) = 0$, i.e. $\phi_1 - \phi_2$ solves the homogeneous equation (3). Therefore one usually needs additional boundary conditions to single out a unique solution $\phi$.

Mathematically, the set of solutions of (5) with fixed $\rho$, $Sol_{i,\rho}$ will be an “affine subspace” of $V$. If $\rho$ is allowed to vary, we have again a linear subspace of solutions, $Sol_i \subset V \oplus W$, which means that $(\phi_1, \rho_1), (\phi_2, \rho_2) \in Sol_i$ and $c_1, c_2 \in \mathbb{C}$ implies

$$(c_1 \phi_1 + c_2 \phi_2, c_1 \rho_1 + c_2 \rho_2) \in Sol_i,$$  \hfill (6)

which means that a superposition principle holds for fields and sources.
2.2 Physical examples and interpretation

2.2.1 Harmonical oscillator

We start with a mechanical example which is also used as a simple model for the interaction between electromagnetic radiation and matter (Lorentz model) to be considered later.

2.2.1.1 The simplest case

Let us first consider the equation of motion of a 1-dimensional harmonical oscillator:

\[ \frac{d^2x}{dt^2}(t) + \omega_0^2 x(t) = 0. \]  (7)

This is a linear, homogeneous equation in the sense considered in the preceding paragraph, if we choose \( \mathcal{V} = \mathcal{W} = \) the space of smooth functions \( x : \mathbb{R} \rightarrow \mathbb{R} \). Here and henceforward we will not be too precise about the exact function spaces appropriate for the various (wave) operators. In the case of (7) the (wave) operator is, of course,

\[ \mathcal{D} = \frac{d^2}{dt^2} + \omega_0^2 . \]  (8)

The superposition principle implies in this case that with the special solutions \( \cos \omega_0 t \) and \( \sin \omega_0 t \) also

\[ x(t) = A \cos \omega_0 t + B \sin \omega_0 t, \quad A, B \in \mathbb{C} \]  (9)

will be a solution of (7). Actually, (9) will be the most general solution. Hence the subspace \( \text{Sol}_h \) of \( \mathcal{V} \) is 2-dimensional.

2.2.1.2 The damped harmonic oscillator with driving force

Next we will consider the equation of motion of a damped harmonic oscillator with driving force:

\[ \frac{d^2x}{dt^2}(t) + 2\Gamma \frac{dx}{dt}(t) + \omega_0^2 x(t) = f(t), \]  (10)

where \( \Gamma > 0 \) is a constant and \( f \) is a general driving force (divided by the mass). This is a linear, inhomogeneous equation with the same \( \mathcal{V} \) and \( \mathcal{W} \) as before and

\[ \mathcal{D} = \frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 . \]  (11)

Again, the superposition principle (6) can be employed to compose the general solution of (10) from special solutions. Let \( G(t) \) denote the special solution of (10) with \( f(t) = \delta(t) \). Since (10) is invariant under time translations, \( G(t - s) \) will also be a solution of (10) with \( f(t) = \delta(t - s) \) for arbitrary \( s \). Then a solution of (10) with general \( f(t) \) can be written as

\[ x(t) = \int_{-\infty}^{\infty} G(t - s)f(s)ds. \]  (12)
G is sometimes called the Greens function of the corresponding (wave) equation. The general solution of (10) is obtained by adding to (12) the general solution of the corresponding homogeneous equation. The latter vanishes for \( t \to \infty \). It remains to find the Greens function for (10). Since \( \delta(t) = 0 \) for \( t > 0 \) it will be a solution of the homogeneous equation for \( t > 0 \) and chosen as \( G(t) = 0 \) for \( t < 0 \) (“causality”). The appropriate initial condition is \( x(0) = 0 \) and \( \dot{x}(0) = 1 \), since the second \( t \)-derivative of this kink gives just \( \delta(t) \). This yields the result

\[
G(t) = \begin{cases} 
1 & \text{for } t > 0 \\
0 & \text{for } t < 0
\end{cases}
\]

for the case \( \Gamma < \omega_0 \). Due to the vanishing of \( G(t-s) \) for \( s > t \) the upper limit of the integral in (12) may be replaced by \( t \).

An important case is a harmonic driving force

\[
f(t) = Ae^{i\omega t}, \ A \in \mathbb{C}.
\]

Due to linearity (and the left hand side of (10) being real) this form can be superposed with \( f(t) = Ae^{-i\omega t} \) to give real \( \sin \omega t \) and \( \cos \omega t \) expressions for \( f(t) \) which have a direct physical interpretation. Instead of solving the integral (12) it is easier in this case to insert the ansatz

\[
x(t) = Be^{i\omega t}, \ B \in \mathbb{C}
\]

into (10) which gives

\[
B = \frac{A}{-\omega^2 + 2i\Gamma \omega + \omega_0^2} \equiv Ar(\omega)e^{i\varphi(\omega)},
\]

\[
r(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2 \omega^2}}: \text{ relative amplitude}
\]

\[
\tan \varphi = \frac{2\Gamma \omega}{\omega^2 - \omega_0^2}, \ \varphi: \text{ phase shift}.
\]

In the case \( \Gamma < \omega_0/\sqrt{2} \) the relative amplitude has its maximum at the value \( \omega_r = \sqrt{\omega_0^2 - 2\Gamma^2} \) which is called the “resonance frequency”. Typical graphs for \( r(\omega) \) and \( \varphi(\omega) \) are shown in the following figures.
Figure 1: Relative amplitude $r$ of forced oscillation as a function of the driving frequency $\omega$.

2.2.2 Electromagnetic waves in a linear medium

2.2.3 Schrödinger equation

2.2.4 Linear dispersion

3 Nonlinear equations

3.1 Generalities

3.2 Nonlinear approximation to linear equations

3.3 Nonlinear 1-dimensional oscillations

3.3.1 Small damping, no driving force

See app. 1.
3.3.2 General case: chaotic motion
See app. 2.

3.3.3 The Duffing oscillator (1918)
See [5]

3.3.3.1 Bistable response  See app. 3.

3.3.3.2 Perturbation series solution  We consider a Duffing oscillator with small cubic nonlinearity and a harmonic driving force. Its equation of motion reads

\[ \ddot{x} + 2\gamma \dot{x} + \alpha x + \lambda x^3 = F \exp(i(\omega t + \phi)) + CC, \]  

(19)

where, as usual, \( CC \) denotes the complex conjugate of the preceding term. It is important to use a real driving force, since the use of complex forces and taking the real part of the resulting solution presupposes linearity. We assume that \( \lambda x^3 \) is small compared with the other terms of (19) and look for solutions which are close to the corresponding
solutions of the linear equation with $\lambda = 0$. Hence it appears natural to expand the solution into a perturbation series, i. e. a power series with respect to $\lambda$:

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \lambda^i.$$  \hspace{1cm} (20)

Inserting this series into the equation of motion (19) and equating terms with the same powers of $\lambda$ yields a hierarchy of linear, inhomogeneous equations which can be explicitly solved. It can be shown by induction that all functions of $t$ which are involved in this hierarchy of equations are periodic functions with frequency $\omega$. Hence they can be expanded into Fourier series, especially

$$x_n(t) = \sum_{m \in \mathbb{Z}} x_{nm} e^{im\omega t}.$$  \hspace{1cm} (21)

Since $x(t)$ is a real function we can write $x_{n,-m} = \overline{x_{nm}}$ and need only consider non-negative $n, m$.

Using computer algebra software it is possible to calculate a certain number of terms of the perturbation series, see app. 4a and 4b. Of course they will become more and more complex.

Here we will only explicitly calculate the first few terms in order to see the underlying principle.

\[ \lambda^0 \text{ terms} \]

It suffices to reproduce the result (16) of the linear problem, which reads, using the slightly different notation of the present section,

$$x_0(t) = x_{01} e^{i\omega t} + C C,$$  \hspace{1cm} (22)

$$x_{01} = h(\omega) F e^{i\phi},$$  \hspace{1cm} (23)

$$h(\omega) \equiv \frac{1}{\alpha - \omega^2 + 2i\gamma \omega}.$$  \hspace{1cm} (24)

\[ \lambda^1 \text{ terms} \]

The cubic term gives

$$\lambda(x_0 + x_1 \lambda + \ldots)^3 = \lambda x_0^3 + \mathcal{O}(\lambda^2) = \lambda(x_{01} e^{3i\omega t} + 3x_{01} x_0 e^{i\omega t} + C C) + \mathcal{O}(\lambda^2).$$  \hspace{1cm} (25)
Since the driving force is \( \mathcal{O}(\lambda^0) \), the \( \mathcal{O}(\lambda^1) \) terms of (19) are

\[
\dot{x}_1 + 2\gamma \dot{x}_1 + \alpha x_1 = -x_0^3 = - \left( x_{11} e^{i\omega t} + 3x_{01} x_{11} e^{i\omega t} + CC \right),
\]

\[
\frac{1}{h(\omega)} x_{11} = -3F^3 e^{i\phi} h^2(\omega) h(\omega),
\]

\[
x_{11} = -3F^3 e^{i\phi} h^3(\omega) h(\omega),
\]

\[
\frac{1}{h(3\omega)} x_{13} = -(F e^{i\phi} h(\omega))^3,
\]

\[
x_{13} = -F^3 e^{3i\phi} h^3(\omega) h(3\omega).
\]

Some remarks are in order. We see in (26) that the cube of the solution in \( \mathcal{O}(\lambda^0) \) approximation acts as a driving force for the next order approximation. Since \( x_0^3 \) contains the frequencies \( \omega \) and \( 3\omega \), also \( x_1(t) \) will contain terms with these frequencies. Both terms are proportional to \( F^3 \), \( F \) being essentially the amplitude of the driving force. Hence in the lowest order the nonlinear term \( \lambda x^3 \) produces a correction to the linear response of the oscillator with the same frequency as well as a correction with triple frequency (third harmonic generation). Both terms could be split into a relative amplitude and a phase shift part, similar as but more complicated than in the linear theory.

\[ \lambda^2 \text{ terms} \]

Similar as above we conclude

\[
\ddot{x}_2 + 2\gamma \dot{x}_2 + \alpha x_2 = -3x_0^2 x_1
\]

\[
= -3 \left( x_{01} e^{i\omega t} + CC \right)^2 \left( x_{11} e^{i\omega t} + x_{13} e^{3i\omega t} + CC \right)
\]

\[
\frac{1}{h(5\omega)} x_{25} = -3x_{01}^2 x_{13}
\]

\[
x_{25} = 3F^5 e^{5i\phi} h^5(\omega) h(3\omega) h(5\omega)
\]

\[
\ldots
\]

It will not be necessary to give more terms. We can already see by the given example that the \( \mathcal{O}(\lambda^2) \) terms will contain the frequencies \( \omega, 3\omega, 5\omega \) and that the corrections will be of order \( F^5 \). We stress this finding in order not to generate the wrong impression that a cubic nonlinearity would at most produce \( 3\omega \) terms proportional to \( F^3 \). At the contrary, it seems fairly clear that the higher order corrections conclude all odd multiples of \( \omega \) and odd powers of \( F \).

4 Cubic Schrödinger equation (CSE)

4.1 Derivation from a Duffing-Lorentz model

The well-known Lorentz model is a simple mechanical model to describe the linear response of matter to electromagnetic waves. The electrons in a piece of matter (insulator)
are considered as classical harmonical damped oscillators subject to the driving force of
the electromagnetic field. The resulting oscillating motion of the electrons produces a
polarization density $P$ which determines the dielectric displacement $D$ and the dielectric
function $\varepsilon(\omega)$. From this a linear wave equation is derived.

In the present subsection we will extend this model to a nonlinear oscillator model with
a cubic nonlinearity $\lambda x^3$ (Duffing oscillator) and consider the resulting correction of
lowest order (see section 3.3.3.2) to the linear wave equation. We call this model the
“Duffing-Lorentz model”. Our treatment is an elaboration of that one sketched in [2],
chapter 16.

We start with the macroscopic Maxwell equations

$$\text{rot} \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = \mathbf{0}, \quad \text{div} \mathbf{D} = \mathbf{0}, \quad (36)$$

$$\text{rot} \mathbf{H} - \frac{\partial}{\partial t} \mathbf{D} = \mathbf{0}, \quad \text{div} \mathbf{B} = \mathbf{0}, \quad (37)$$

and assume the relations

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad (38)$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}). \quad (39)$$

Taking the curl of (36) and using (37), (38) and (39) yields

$$\text{rot} \text{rot} \mathbf{E} = \nabla (\nabla \cdot \mathbf{E} - \Delta \mathbf{E}) \quad (40)$$

$$= -\frac{\partial}{\partial t} \nabla \times \mathbf{B} \quad (41)$$

$$= -\mu_0 \frac{\partial^2}{\partial t^2} \mathbf{D} \quad (42)$$

$$= -\mu_0 \left( \varepsilon_0 \ddot{\mathbf{E}} + \dot{\mathbf{P}} \right). \quad (43)$$

Next we assume the special situation where the fields $\mathbf{E}$ and $\mathbf{P}$ points into $z$-direction
and depends only on $x, y, t$. Let us write

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ E(x, y, t) \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ P(x, y, t) \end{pmatrix}. \quad (44)$$

It follows that $\nabla \cdot \mathbf{E} = 0$ and

$$\Delta \mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ E_{xx} + E_{yy} \end{pmatrix}, \quad (45)$$

hence (43) reduces to the scalar equation

$$E_{xx} + E_{yy} = \mu_0 \left( \varepsilon_0 \ddot{E} + \dot{P} \right). \quad (46)$$
The Duffing-Lorentz model is defined by the equation of motion for a typical electron with mass $m$ and charge $q$

$$m \ddot{z} + 2 \Gamma \dot{z} + maz + \Lambda z^3 = q \tilde{E} e^{i\omega t} + CC. \quad (47)$$

Division by the electron’s mass and introducing new parameters yields the form of the equation of motion we have already considered before, see (19),

$$\ddot{z} + 2 \gamma \dot{z} + \alpha z + \lambda z^3 = F e^{i\phi} e^{i\omega t} + CC. \quad (48)$$

According to (22) - (30) the nonlinear response to the harmonic driving force will be

$$z(t) = \left( Fe^{i\phi} h(\omega)e^{i\omega t} - 3\lambda F^3 e^{i\phi} h^2(\omega)\overline{h}(\omega)e^{i\omega t} - \lambda F^3 e^{i\phi} h^3(\omega)h(3\omega)e^{3i\omega t} \right) + CC \quad (49)$$

This gives rise to an electric dipole density

$$P = qNz(t), \quad (50)$$

where $N$ denotes the constant electron density.

In the approximation we consider there are only Fourier components of $E$ and $P$ with the frequencies $\omega$ and $3\omega$, i.e. we have

$$E = (E_1(x, y)e^{i\omega t} + E_3(x, y)e^{3i\omega t}) + CC, \quad (51)$$

$$P = (P_1(x, y)e^{i\omega t} + P_3(x, y)e^{3i\omega t}) + CC. \quad (52)$$

Thus (46) assumes the form

$$E_{1xx} + E_{1yy} = \mu_0 (\varepsilon_0 E_1 + P_1)(-\omega^2), \quad (53)$$

$$E_{3xx} + E_{3yy} = \mu_0 (\varepsilon_0 E_3 + P_3)(-9\omega^2). \quad (54)$$

Using the identification $\frac{qE_1}{m} = Fe^{i\phi}$ we obtain by (49) and (50)

$$P_1 = qN \left( \frac{qE_1}{m} h(\omega) - 3\lambda \left( \frac{q}{m} \right)^3 E_1 |E_1|^2 h^2(\omega)\overline{h}(\omega) \right) \quad (55)$$

$$= \frac{1}{\mu_0} \left( n_1(\omega) E_1 + n_3(\omega) E_1 |E_1|^2 \right), \quad (56)$$

$$P_3 = qN \left( -\lambda \left( \frac{q}{m} \right)^3 h^3(\omega)h(3\omega) + \frac{qE_3}{m} h(3\omega) \right). \quad (57)$$

If (56) is inserted into (53) we obtain the nonlinear wave equation

$$E_{1xx} + E_{1yy} = -\omega^2 \left( (\varepsilon_0 \mu_0 + n_1(\omega)) E_1 - n_3(\omega) E_1 |E_1|^2 \right). \quad (58)$$

If its solution is inserted into (57), (54) assumes the form of a linear inhomogeneous wave equation for the $3\omega$ component of the electromagnetic wave. We will not further
consider the latter. Returning to (58) we will make the ansatz of a modulated plane wave in $x$-direction with an amplitude envelope

$$E_1(x, y) = \psi(x, y) \exp(-ikx)$$  \hspace{1cm} (59)$$

and obtain

$$E_{1x} = (\psi_x - ik\psi) \exp(-ikx),$$  \hspace{1cm} (60)$$

$$E_{1xx} = (\psi_{xx} - 2ik\psi_x - k^2\psi) \exp(-ikx),$$  \hspace{1cm} (61)$$

$$E_{1yy} = \psi_{yy} \exp(-ikx).$$  \hspace{1cm} (62)$$

Hence (58) assumes the form

$$\left(\psi_{xx} + \psi_{yy} - 2ik\psi_x - k^2\psi\right) \exp(-ikx) = -\omega^2(\varepsilon_0\mu_0 + n_1(\omega))\psi \exp(-ikx)$$

$$-\omega^2 n_3(\omega)\psi |\psi|^2 \exp(-ikx).$$  \hspace{1cm} (63)$$

Using

$$\omega^2 = \varepsilon_0\mu_0 k^2$$  \hspace{1cm} (64)$$

and the “thin beam approximation”

$$\psi_{xx} \ll \psi_{yy}$$  \hspace{1cm} (65)$$

we arrive at the wave equation

$$-2ik\psi_x + \psi_{yy} + \omega^2 n_1(\omega)\psi + \omega^2 n_3(\omega)|\psi|^2\psi = 0.\hspace{1cm} (66)$$

The term $\omega^2 n_1(\omega)\psi$ can be made to vanish by means of a transformation

$$\Psi = \psi \exp(-icx),$$  \hspace{1cm} (67)$$

if we chose $2kc = \omega^2 n_1(\omega)$. Hence we arrive at a cubic Schrödinger equation of the form

$$-2ik\Psi_x + \Psi_{yy} + \omega^2 n_3(\omega)|\Psi|^2\Psi = 0\hspace{1cm} (68)$$

which is equivalent to the above-mentioned standard form

$$-i\Psi_t + \frac{1}{2}\Psi_{xx} + \sigma|\Psi|^2\Psi = 0\hspace{1cm} (69)$$

after a suitable scaling and re-naming of the variables. However, these transformations presuppose that $n_1(\omega)$ and $n_3(\omega)$ are real functions. This is only approximately satisfied if the damping coefficient $\gamma$ of the Duffing-Lorentz model can be neglected. This is a consistent assumption: In the next subsections we will see that the CSE satisfies an energy conservation law, which can only be valid if dissipative effects are neglected.
4.2 General properties

We will first collect some simple properties of the CSE

\[ i\psi_t + \frac{1}{2}\psi_{xx} + \sigma|\psi|^2\psi = 0. \]  (70)

Let \( \mathcal{CSE} \) denote the set of solutions of (70) subject to the boundary conditions that \( \psi \) as well as its arbitrary \( x \)-derivatives vanish at \( x \to \pm \infty \). Then the following properties of \( \mathcal{CSE} \) hold:

- Linearity with respect to phase factors
  \[ \psi \in \mathcal{CSE} \text{ and } \beta \in \mathbb{R} \Rightarrow e^{i\beta}\psi \in \mathcal{CSE}. \]  (71)

- Time reversal invariance
  \[ \psi \in \mathcal{CSE} \text{ and } \phi(x, t) = \overline{\psi(x, -t)} \Rightarrow \psi \in \mathcal{CSE}. \]  (72)

- Scaling property
  \[ \psi \in \mathcal{CSE} \text{ and } \phi(x, t) = \alpha\psi(\alpha x, \alpha^2 t), \alpha \in \mathbb{R} \Rightarrow \phi \in \mathcal{CSE}. \]  (73)

- Translational invariance
  \[ \psi \in \mathcal{CSE} \text{ and } \phi(x, t) = \psi(x + a, t + \tau), a, \tau \in \mathbb{R} \Rightarrow \phi \in \mathcal{CSE}. \]  (74)

- Galilean invariance
  \[ \psi \in \mathcal{CSE} \text{ and } \phi(x, t) = \exp(-\frac{i}{2}v^2 t + ivx)\psi(x - vt, t), v \in \mathbb{R} \Rightarrow \phi \in \mathcal{CSE}. \]  (75)

We will now sketch the proofs of these properties. Obviously, (71) holds since multiplying \( \psi \) by \( e^{i\beta} \) results in multiplying the whole CSE by an arbitrary phase factor.

(72) follows by taking the complex conjugate of the CSE and using \( -i\psi_t = i\phi_t \).

To prove the scaling property we set \( \phi(x, t) = \gamma\psi(\alpha x, \beta t) \) and consider the following terms of the CSE

\[
\begin{align*}
\phi_t(x, t) & = \beta\gamma\psi_t(\alpha x, \beta t), \\
\phi_{xx}(x, t) & = \alpha^2\gamma\psi_{xx}(\alpha x, \beta t), \\
|\phi|^2\phi & = \gamma^2\gamma|\psi(\alpha x, \beta t)|^2\psi(\alpha x, \beta t).
\end{align*}
\]  (76)\hspace{1cm} (77)\hspace{1cm} (78)

All three terms obtain the same factor if \( \beta = \alpha^2 \) and \( \alpha = \gamma \).

Translational invariance (74) holds since the coefficients of the CSE are not depending on \( x \) or \( t \).
It remains to show Galilean invariance of the CSE. The proof is completely analogous to the quantum mechanical case. We set
\[ \phi(x,t) = \exp(i\lambda t + ikx)\psi(x - vt, t) \equiv e^\psi \]
and obtain
\[ \phi_t = i\lambda e^\psi + e^\psi_t - ve^\psi_x, \]
\[ \phi_x = ike^\psi + e^\psi_x, \]
\[ \psi_{xx} = -k^2 e^\psi + 2ike^\psi_x + e^\psi_{xx}. \]
Hence
\[ \phi_t + \frac{1}{2} \phi_{xx} + \sigma |\phi|^2 \phi = -ive^\psi_x + ike^\psi_x - \lambda e^\psi - \frac{1}{2} k^2 e^\psi + e \left( i\psi_t + \frac{1}{2} \psi_{xx} + \sigma |\psi|^2 \phi \right). \]
We see that \( \phi \in \mathcal{CSE} \Leftrightarrow \psi \in \mathcal{CSE} \) if the first 4 terms of the r. h. s. of (83) vanish. This is achieved for \( v = k \) and \( \lambda = -\frac{1}{2} k^2 \) which concludes the proof of (75).

4.3 Conservation laws

The CSE as well as the KdV equation possess an infinite number of conserved quantities. This is the crucial property which leads to interpreting these equations as infinite dimensional integrable Hamiltonian systems. In this course we will only treat the first three conservation laws of the CSE which can be confirmed by elementary calculations.

We will write the CSE and its complex conjugate in the form
\[ \psi_t = \frac{i}{2} \psi_{xx} + i\sigma \overline{\psi} \psi^2, \]
\[ \overline{\psi}_t = -\frac{i}{2} \overline{\psi}_{xx} - i\sigma \psi \overline{\psi}^2, \]
and will make frequent use of integration by parts where the boundary terms always vanish due to our boundary conditions for \( x \to \pm \infty \). Recall from quantum mechanics that the scalar product of wave functions is defined by
\[ \langle \phi | \psi \rangle \equiv \int_R \overline{\phi(x)} \psi(x) dx. \]
The first conserved quantity to be considered is \( \langle \psi | \psi \rangle \). In quantum mechanics \( (\sigma = 0) \) this is the total probability \((= 1)\) which should be conserved in the course of time. In the context of nonlinear optics \( \langle \psi | \psi \rangle \) is proportional to the field energy and its conservation is plausible as long as dissipative effects are neglected. In order to prove
\[ \frac{d}{dt} \langle \psi | \psi \rangle = 0 \]
we conclude
\[
\frac{d}{dt} \langle \psi | \psi \rangle = \langle \psi | \dot{\psi} \rangle + \langle \dot{\psi} | \psi \rangle \tag{88}
\]
\[
= \langle \frac{i}{2} \psi_{xx} + i\sigma |\psi|^2 \psi | \psi \rangle + \langle \psi | \frac{i}{2} \psi_{xx} + i\sigma |\psi|^2 \psi \rangle \tag{89}
\]
\[
= -\frac{i}{2} \left( \int \overline{\psi} \psi_{xx} dx - \int \overline{\psi} \psi_{xx} dx \right) \tag{90}
\]
\[
+ \sigma \left( \int (-i\overline{\psi} |\psi|^2 \psi dx + \int (i\overline{\psi} |\psi|^2 \psi dx \right) \tag{91}
\]
\[
= -\frac{i}{2} \left( -\int \overline{\psi} \psi_{x} dx + \int \overline{\psi} \psi_{x} dx \right) = 0. \tag{92}
\]

In the last step we used integration by parts.

In quantum mechanics the expectation value of momentum \(\langle \psi | \mathcal{P} | \psi \rangle = \frac{\hbar}{i} \langle \psi | \psi_x \rangle\) would be conserved for a free particle. Hence one may conjecture that also for solutions of the CSE we have

\[
\frac{d}{dt} \int \overline{\psi} \psi_x = 0. \tag{93}
\]

The proof is again straight forward:
\[
\frac{d}{dt} \int \overline{\psi} \psi_x = \int \psi \overline{\psi_x} dx + \int \psi \overline{\psi_{xt}} \tag{94}
\]
\[
= i \int \left( \frac{1}{2} \overline{\psi_{xx}} + \sigma \overline{\psi^2} \right) \psi_x dx - i \int \left( \frac{1}{2} \overline{\psi_{xxx}} + \sigma \overline{\psi^2} \right) \psi dx \tag{95}
\]
\[
= i \int \left( -\frac{1}{2} \overline{\psi_x} \overline{\psi_{xx}} + \sigma \overline{\psi^2} \overline{\psi_x} + \frac{1}{2} \overline{\psi_{xx}} \overline{\psi_x} + \sigma \overline{\psi^2} \psi \right) dx \tag{96}
\]
\[
= \frac{i\sigma}{2} \int \left( \overline{\psi^2} \overline{\psi^2} \right) dx = 0. \tag{97}
\]

As in quantum mechanics one may conclude from (93) that the suitable defined “center of mass” of the wave packet moves with uniform velocity. We will leave the details to the reader.

The next conserved quantity in analogy with quantum mechanics would be the total kinetic energy expectation value \(\langle \psi | \mathcal{P}^2 | \psi \rangle\). However, it turns out that for \(\sigma \neq 0\) we have
to add a bi-quadratic term in order to obtain a conserved quantity:

\[
\frac{d}{dt} \int (\psi_x \overline{\psi}_x - \sigma \psi^2 \overline{\psi}^2) dx
\]

(98)

\[
i \int \left( \psi_{tx} \overline{\psi}_x + \psi_x \overline{\psi}_{tx} - 2 \sigma \psi_t \overline{\psi}^2 - 2 \sigma \psi^2 \overline{\psi}_x \psi_x \right) dx
\]

(99)

\[
i \int \left\{ \left( \frac{1}{2} \psi_{xxx} + \sigma (\overline{\psi} \psi^2)_x \right) \overline{\psi}'_x - \left( \frac{1}{2} \overline{\psi}_{xxx} + \sigma (\overline{\psi} \psi^2)_x \right) \psi_x \right\} dx
\]

(100)

\[
-2\sigma \left( \frac{1}{2} \psi_{xx} + \sigma \overline{\psi} \psi^2 \right) \overline{\psi}'^2 - \left( \frac{1}{2} \overline{\psi}_{xx} + \sigma \overline{\psi} \psi^2 \right) \overline{\psi}'^2 \right\} dx
\]

(101)

= 0.

(102)

In the last step the 8 terms cancel pairwise due to integration by parts and according to the powers of \( \sigma \).

### 4.4 1-soliton solutions

In order to derive these special solutions we insert the ansatz

\[
\psi(x, t) = \phi(x) e^{i\alpha t}, \ \alpha \in \mathbb{R}
\]

(103)

into the CSE (70) and obtain

\[-\alpha \phi + \frac{1}{2} \phi_{xx} + \sigma \phi^3 = 0.\]

(104)

In analogy with the treatment of 1-dimensional equations of motion we multiply with \( 4\phi_x \) and integrate over \( x \) which gives

\[\phi_x^2 - 2\alpha \phi^2 + \sigma \phi^4 = C.\]

(105)

The straight forward solution procedure would be solving (105) for \( \phi_x \) followed by a separation of variables which leads to the elliptical integral

\[
\int \frac{d\phi}{\sqrt{C - \sigma \phi^4 + \alpha \phi^2}} = \int dx.
\]

(106)

For \( \alpha, \sigma > 0, \ C < 0 \) we obtain periodic solutions. Soliton solutions occur for \( \alpha, \sigma > 0, \ C = 0 \).

Instead of solving the above integral (106) one can perform the transformation \( \phi(x) = f^{-1}(x) \). Using \( \phi_x = -f^{-2} f_x \) equation (105) is transformed into

\[f^{-4} f_x^2 - 2\alpha f^{-2} + \sigma f^{-4} = 0.\]

(107)

Multiplication with \( f^4 \), differentiation with respect to \( x \) and division by \( 2f_x \) yield

\[f_x^2 - 2\alpha f^2 + \sigma = 0\]

(108)

\[2f_x f_{xx} - 4 f f_x = 0\]

(109)

\[f_{xx} - 2\alpha f = 0.\]

(110)
The general solution of (110) which is compatible with the above boundary conditions reads

\[ f(x) = A \cosh(\sqrt{2\alpha}(x - x_0)). \]  

(111)

Since the CSE is invariant with respect to \( x \)-translations we can, without loss of generality, set \( x_0 = 0 \). We have still to check whether (111) satisfies (108), since by differentiation of an differential equation we might have enlarged the set of solutions. Using

\[ f_x = A\sqrt{2}\alpha \sinh(\sqrt{2}\alpha x), \]  

(112)

we obtain

\[ -f_x^2 + 2\alpha f^2 = 2\alpha A^2(\cosh^2(\sqrt{2}\alpha x) - \sinh^2(\sqrt{2}\alpha x)) = 2\alpha A^2 = \sigma. \]  

(113)

Hence the parameters \( A, \alpha, \sigma \) cannot be chosen independently but have to satisfy

\[ A = \sqrt{\frac{\sigma}{2\alpha}}. \]  

(114)

Inverting the above transformation we obtain

\[ \psi(x) = \sqrt{\frac{2\alpha}{\sigma}} \sech(\sqrt{2\alpha}x)e^{i\alpha t}. \]  

(115)

Finally, we may apply a Galileo transformation (75) to (115) which yields

\[ \Psi(x, t) = \sqrt{\frac{2\alpha}{\sigma}} \exp\left(i(\alpha - \frac{1}{2}v^2)t + ivx\right) \sech\left(\sqrt{2\alpha}(x - vt)\right). \]  

(116)

This represents a soliton moving with velocity \( v \) in \( x \)-direction without changing its form, see figure 3. Its amplitude is inversely proportional to its width. In fact, it is not difficult to show that the distance \( d \) between the two points of inflection of the soliton profile amounts to

\[ d = \sqrt{\frac{2}{\alpha}} \log(\sqrt{2} + 1). \]  

(117)

### 4.5 Interaction of 2 solitons

In this subsection we provide some numerical and analytical studies of the interaction of two solitons. The analytical solutions are taken from [4] and obtained by methods to be explained later, see section 6. The main result is that two solitons which approach with opposed velocities, interact and then fly apart without changing their form. The only result of the scattering is a shift in position and in phase, see app. 5.
Figure 3: 1-soliton solution of the cubic Schrödinger equation.

5 Korteweg-de Vries equation

The KdV equation has first been derived in 1895 by D. J. Korteweg and G. de Vries in order to describe solitary water waves first observed by J. S. Russell in 1834, see e. g. [3], p. 1.

5.1 1-soliton solutions

A normalized form of the KdV equation reads

\[ q_t + \sigma q q_x + q_{xxx} = 0, \quad \sigma > 0. \]  

(118)

We are mainly interested in real solutions. The KdV equation admits 1-soliton solutions analogously to those derived for the cubic Schrödinger equation in section 4.4. They are of the form

\[ q(x, t) = s(x - vt), \quad v > 0. \]  

(119)
This represents a wave travelling with velocity \( v \) to the right without changing its shape. The ansatz (119) implies \( q_t = -v s_x, q_x = s_x \), etc.. Inserting this into (118) yields

\[
-v s_x + \sigma s s_x + s_{xxx} = 0. \tag{120}
\]

Integrating over \( x \) gives

\[
-v s + \frac{\sigma}{2} s^2 + s_{xx} = C. \tag{121}
\]

The constant of integration \( C \) vanishes if we impose the boundary conditions that \( s \) as well as its derivatives \( s_x, s_{xx} \) vanish for \( |x| \to \infty \), which is a natural condition for soliton-like solutions.

Note that (121) is formally identical with the one-dimensional equation of motion of a unit mass with a force law \( F(s) = v s - \frac{\sigma}{2} s^2 \), if \( s(x) \) is construed as the position \( s \) at time \( x \). Following the usual procedure of solving such one-dimensional problems we multiply (121) with \( 2s_x \) and once more integrate over \( x \). This yields the “energy conservation law”

\[
-v s^2 + \frac{\sigma}{3} s^3 + s_x^2 = \tilde{C} = 0. \tag{122}
\]

Again the “total energy” \( \tilde{C} \) vanishes due to the boundary conditions. The straightforward solution procedure would be solving (122) for \( s_x \) followed by a separation of variables which leads to the solvable integral

\[
\int \frac{ds}{s \sqrt{v - \sigma s/3}} = \int dx. \tag{123}
\]

Alternatively one can perform the transformation \( s(x) = f^{-2}(x) \). Using \( s_x = -2 f^{-3} f_x \) equation (122) is transformed into

\[
-v f^{-4} + \frac{\sigma}{3} f^{-6} + 4 f^{-6} f_x^2 = 0. \tag{124}
\]

Multiplication with \( f^6 \), differentiation with respect to \( x \) and division by \( 2 f_x \) yield

\[
-v f^2 + \frac{\sigma}{3} + 4 f_x^2 = 0 \tag{125}
\]

\[
-2 v f f_x + 8 f_x f_{xx} = 0 \tag{126}
\]

\[
f_{xx} - \frac{v}{4} f = 0. \tag{127}
\]

The general solution of (127) which is compatible with the above boundary conditions reads

\[
f(x) = A \cosh(\frac{\sqrt{v}}{2} (x - x_0)). \tag{128}
\]

Note that for the last step we used the first time the restriction \( v > 0 \). A negative \( v \) would give oscillating solutions which wouldn’t look like solitons. Since the KdV
equation is invariant with respect to \( x \)-translations we can, without loss of generality, set \( x_0 = 0 \). We have still to check whether (128) satisfies (125), since by differentiation of a differential equation we might have enlarged the set of solutions. Using

\[
 f_x = \frac{\sqrt{v}}{2} A \sinh\left(\frac{\sqrt{v}}{2} x\right). 
\]  

(129)

we obtain

\[
 -4f_x^2 + vf^2 = vA^2 (\cosh^2(\frac{\sqrt{v}}{2} x) - \sinh^2(\frac{\sqrt{v}}{2} x)) = vA^2 = \frac{\sigma}{3}. 
\]  

(130)

Hence the parameters \( A, v, \sigma \) cannot be chosen independently but have to satisfy

\[
 A = \sqrt{\frac{\sigma}{3v}}. 
\]  

(131)

Inverting the above transformations we obtain

\[
 s(x) = \frac{3v}{\sigma} \text{sech}^2\left(\frac{\sqrt{v}}{2} x\right) 
\]  

(132)

and finally

\[
 q(x, t) = \frac{3v}{\sigma} \text{sech}^2\left(\frac{\sqrt{v}}{2} (x - vt)\right). 
\]  

(133)

We have obtained a 1-parameter family of soliton solutions. The parameter \( v \), which was originally introduced as the velocity of the soliton, also determines the form of the wave profile: The amplitude is proportional to \( v \) and the width of the soliton is proportional to \( \frac{1}{\sqrt{v}} \). In fact, it is not difficult to show that the distance \( d \) between the two points of inflection of the wave profile amounts to

\[
 d = \ln \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right) \frac{2}{\sqrt{v}}, 
\]  

(134)

see figure 4. This is typical for KdV-solitons whereas for CSE-solitons the amplitude and the velocity are independent, see section 4.3.

5.2 A physical derivation of the KdV equation for water waves

We will present a derivation of a certain variant of the KdV equation from first principles following closely the account of G. B. Whitham in [2], chapter 13.

We consider the two-dimensional hydrodynamic wave problem. The geometry is explained in figure 5:

A solitary wave train is moving into \( x \)-direction. The height of the water at rest is \( h \), the solitary wave has a typical height \( a \) and length \( \ell \). For the sake of reference our
assumptions which will lead to the KdV equation are contained in table 1.

According to assumption 1 the velocity field of the fluid can be written in the form

\[
\vec{v}(x, y, t) = \left( \begin{array}{c} u(x, y, t) \\ v(x, y, t) \end{array} \right).
\] (135)

Due to the assumption 2 of constant mass density \( \rho \) the usual equation of mass conservation (analogous to the continuity equation in electrodynamics)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\] (136)

assumes the simplified form:

\[
0 = \nabla \cdot \vec{v} = u_x + v_y.
\] (137)

For an irrotational flow (assumption 3) the velocity field can be written as the gradient of a "velocity potential" \( \varphi \):
Figure 5: Geometry of the hydrodynamic wave problem which leads to the Korteweg-de Vries equation.

\[
\nabla \times \vec{v} = \vec{0} \quad \Rightarrow \quad \vec{v} = \nabla \phi
\]

\[
u = \phi_x, \quad v = \phi_y.
\]  \hspace{1cm} (138)

(137) and (138) imply Laplace’s equation

\[
\Delta \phi = \phi_{xx} + \phi_{yy} = 0.
\]  \hspace{1cm} (139)

Although (139) is a linear equation its boundary conditions will be nonlinear, as we will see in a minute. According to assumption 4 (a) the vertical velocity component \( v \) will vanish at \( y = 0 \) for all \( x \) and \( t \):

\[
v(x, 0, t) = \phi_y(x, 0, t) = 0.
\]  \hspace{1cm} (140)

The remaining boundary conditions refer to the free water surface which is not fixed but evolves in time according to the wave equation. Thus we are faced with what is usually called a “free boundary problem”. We will write the equation for the free water surface in the form:

\[
y = \eta(x, t) \equiv h + q(x, t).
\]  \hspace{1cm} (141)
Table 1: Assumptions leading to a wave equation of KdV type.

Assumptions

1. 2-dimensional problem, no \( z \)-dependence
2. Constant mass density of water
3. Irrotational flow
4. Boundary conditions:
   a) No flow across the river-bed
   b) No flow across the free surface
   c) Constant pressure at the free surface
5. Ideal fluid (viscosity is neglected)
6. Limit case
   a) \( h \ll \ell \)
   b) \( a \ll h \)
7. The solitary wave(s) move(s) to the right

The function \( \eta \) (or \( q \)) can be viewed as an additional unknown which is linked to the other unknowns such as \( \varphi \) via the boundary conditions. According to assumption 4 (b) a mass element at the surface remains at the surface for all times. Its coordinates hence satisfy

\[
y(t) = \eta(x(t), t).
\]  

(142)

The total \( t \)-derivative of (142) yields

\[
v = \frac{dy}{dt} = \frac{\partial \eta}{\partial x} u + \frac{\partial \eta}{\partial t},
\]  

(143)

or

\[
\varphi_y = \frac{\partial \eta}{\partial x} \varphi_x + \frac{\partial \eta}{\partial t}.
\]  

(144)

Of course, (144) only holds at the free surface, i.e. for \( y = \eta(x, t) \). The remaining boundary condition 4 (c) refers to the pressure \( p \) which is introduced by the equations expressing momentum conservation (or balance of forces). These equations are called “Navier-Stokes-equations” and simplify to “Euler’s equation” for ideal fluids (assumption 5):

\[
\frac{d\vec{\nu}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p - \left( \begin{array}{c} 0 \\ g \end{array} \right),
\]  

(145)
where \( p(x, y, t) \) is the fluid’s pressure and \( g \) is the gravitational acceleration. Since \( \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) = (\vec{v} \cdot \nabla)\vec{v} + \vec{v} \times (\nabla \times \vec{v}) \) and \( \nabla \times \vec{v} = \vec{0} \) by (138), we conclude

\[
\frac{\partial}{\partial t} \nabla \varphi + \nabla \left( \frac{1}{2} |\vec{v}|^2 \right) + \nabla \left( \frac{p}{\rho} + gy \right) = \vec{0},
\]

(146)

and hence

\[
\varphi_t + \frac{1}{2} (\nabla |\vec{v}|)^2 + \frac{p}{\rho} + gy = C.
\]

(147)

Assumption 4 (c) means that (147) holds with \( p = p_0 \) at the free surface. Differentiating with respect to \( x \) finally yields

\[
u_t + uu_x + vv_x + g\eta_x = 0.
\]

(148)

Next we will expand the velocity potential into a power series with respect to \( y \). At the moment, this is to be considered as a purely formal procedure; later on we will argue that the first few terms of this series yield a reasonable approximation of \( \varphi \). Thus let

\[
\varphi(x, y, t) = \sum_{n=0}^{\infty} y^n \varphi_n(x, t).
\]

(149)

The velocity potential satisfies Laplace’s equation, see (139). Hence from

\[
\frac{\partial^2}{\partial x^2} \varphi = \sum_{n=0}^{\infty} y^n \frac{\partial^2}{\partial x^2} \varphi_n(x, t),
\]

(150)

\[
\frac{\partial^2}{\partial y^2} \varphi = \sum_{n=2}^{\infty} n(n-1)y^{n-2}\varphi_n(x, t)
\]

(151)

\[
= \sum_{n=0}^{\infty} (n+2)(n+1)y^n\varphi_{n+2}(x, t)
\]

(152)

we conclude the following recursion relation for \( \varphi_n \):

\[
\frac{\partial^2}{\partial x^2} \varphi_n(x, t) + (n + 1)(n + 2)\varphi_{n+2}(x, t) = 0, \quad \text{for all } n \in \mathbb{N}.
\]

(153)

The first boundary condition (140) yields

\[
0 = \left. \frac{\partial}{\partial y} \varphi(x, y, t) \right|_{y=0} = \sum_{n=1}^{\infty} ny^{n-1}\varphi_n(x, t)|_{y=0} = \varphi_1(x, t).
\]

(154)

25
Hence (154) yields

\[ n = 0 : \quad \frac{\partial^2}{\partial x^2} \varphi_0 + 2 \cdot 1 \cdot \varphi_2 = 0 \quad (155) \]

\[ n = 1 : \quad \frac{\partial^2}{\partial x^2} \varphi_1 + 3 \cdot 2 \cdot \varphi_3 = 0 \quad (156) \]

\[ \Rightarrow \varphi_3(x, t) = 0 \quad (157) \]

\[ n = 2 : \quad \frac{\partial^2}{\partial x^2} \varphi_2 + 4 \cdot 3 \cdot \varphi_4 = 0 \quad (158) \]

\[ n = 3 : \quad \frac{\partial^2}{\partial x^2} \varphi_3 + 5 \cdot 4 \cdot \varphi_5 = 0 \quad (159) \]

\[ \Rightarrow \varphi_5(x, t) = 0 \quad (160) \]

\[ \ldots \quad (161) \]

Thus all odd \( \varphi_n \) vanish and the even \( \varphi_n \) can be expressed by even \( x \)-derivatives of \( \varphi_0 \):

\[ \varphi(x, y, t) = \sum_{n=0}^{\infty} y^{2n} (-1)^n \frac{1}{(2n)!} \left( \frac{\partial}{\partial x} \right)^{2n} \varphi_0(x, t). \quad (162) \]

According to assumption 6 we introduce the small parameters

\[ \alpha \equiv \frac{a}{h} \ll 1, \quad \beta \equiv \frac{h^2}{\ell^2} \ll 1, \quad (163) \]

which are assumed to be of the same order of magnitude and analyze the wave problem in the asymptotic limit \( \alpha, \beta \to 0 \). For this goal we transform our variables and equations into a dimensionless form. After this transformation we denote all variables with the same letters as before, hence it amounts to the simultaneous substitution

\[ x \to \ell x \quad (164) \]

\[ y \to hx \quad (165) \]

\[ t \to \frac{\ell}{c} t \quad (166) \]

\[ q \to aq \quad (167) \]

\[ \varphi \to \frac{g \ell a}{c} \varphi, \quad (168) \]

where

\[ c \equiv \sqrt{gh} \quad (169) \]

denotes a typical velocity. It can be shown to equal the velocity of linear water waves in lowest order of approximation. It is crucial for this approach that there is a different stretching in \( x \)- and in \( y \)-direction. The chosen units which lead to the transformation are not unique due to the appearance of two dimensionless quantities \( \alpha \) and \( \beta \). Any choice of units different from the present one would lead to a different asymptotic limit.
of the wave equation.

We now consider the transformed equations. The dimensionless equation of the free surface will be

\[ y = \eta(x, t) = 1 + \alpha q(x, t). \]  

(170)

Equation (144) is equivalent to

\[ \varphi_y = q_x \varphi_x + q_t. \]  

(171)

After substituting the dimensionless variables we obtain

\[ \frac{gla}{c} \frac{\varphi}{y} = \frac{a gla}{c} \frac{q_x}{\varphi_x} + \frac{ac}{\ell} q_t. \]  

(172)

Multiplication with \( \frac{\ell}{ac} \) finally leads to

\[ \frac{1}{\beta} \varphi_y = \alpha q_x \varphi_x + q_t. \]  

(173)

Analogously, (148) is transformed into the dimensionless form

\[ \varphi_{xt} + \alpha \varphi_x \varphi_{xx} + \frac{\alpha}{\beta} \varphi_y \varphi_{yx} + q_x = 0. \]  

(174)

The power series expansion of \( \varphi \) (162) reads in dimensionless form

\[ \varphi(x, y, t) = \sum_{n=0}^{\infty} y^{2n} \frac{(-1)^n}{(2n)!} \beta^n \left( \frac{\partial}{\partial x} \right)^{2n} \varphi_0(x, t) \]  

(175)

\[ = \varphi_0(x, t) - \frac{\beta}{2} y^2 \varphi_{0xx}(x, t) + \frac{\beta^2}{24} y^4 \varphi_{0xxxx}(x, t) + \ldots \]  

(176)

We note that it can be viewed as a power series with respect to the small parameter \( \beta \) which justifies the use of the first few terms as an approximation. More precisely, we will insert (176) into the dimensionless boundary conditions (173) and (174), using (170) and retain only constant or linear terms with respect to \( \alpha \) and \( \beta \). Here the assumption that \( \alpha \) and \( \beta \) are of the same order of magnitude is crucial. In what follows we will use the abbreviation

\[ w(x, t) \equiv \varphi_{0x}(x, t). \]  

(177)

After some calculations the boundary conditions assume the asymptotic form

\[ w_x + q_t + \alpha(qw)_x - \frac{\beta}{6} w_{xxx} = \mathcal{O}_2 \]  

(178)
\[ w_t + q_x + \alpha w w_x - \frac{\beta}{2} w_{xxt} = \mathcal{O}_2, \tag{179} \]

where \( \mathcal{O}_2 \) stands for terms of quadratic and higher order in \( \alpha, \beta \).

In lowest order we consider only constant terms. Then (178) and (179) reduce to

\[ w_x = -q_t, \quad w_t = -q_x. \tag{180} \]

Differentiation with respect to \( x \) leads to

\[ w_{xx} = -q_{tx} = -q_{xt} = w_{tt}, \tag{181} \]

and analogously for \( q \)

\[ q_{xx} = q_{tt}. \tag{182} \]

These are 1-dimensional linear wave equations for \( w \) and \( q \) with the dimensionless wave velocity 1. This confirms the above remark about the physical meaning of the typical velocity \( c \). The general solutions of (181) and (182) are well-known: Travelling wave trains of the form

\[ w(x, t) = f(x \pm t), \quad q(x, t) = F(x \pm t), \tag{183} \]

with arbitrary smooth functions \( f \) and \( F \). Using the above assumption 7, that \( q \) is travelling to the right, we conclude \( q(x, t) = F(x - t) \). Since

\[ \begin{align*}
    w_x &= f'(x \pm t) = -q_t = F'(x - t), \\
    w_t &= \pm f'(x \pm t) = -q_x = -F'(x - t),
\end{align*} \tag{184} \tag{185} \]

the only consistent solutions of (181) and (182) under assumption 7 will be of the form

\[ w(x, t) = f(x - t), \quad f(\xi) = F(\xi) + C. \tag{186} \]

The constant \( C \) vanishes since \( q \) and \( w \) vanish for \( |\xi| \to \infty \), hence in lowest order approximation

\[ w(x, t) = q(x, t) = F(x - t). \tag{187} \]

We return to the linear order approximation. (178) and (179) represent two coupled, nonlinear partial differential equations. The straightforward approach to decouple these equations would be differentiating and using \( q_{xt} = q_{tx} \). But this leads to equations of second order in the time derivative, whereas the KdV equation is of first order. Therefore we will use a different strategy. Recall that (178) and (179) are first order approximations of a more complex system of equations. Thus it would be legitimate to insert the result (187) of the zeroth order approximation into the first order terms of (178) and
since the difference is at least of quadratic order.

For this reason we may repeatedly use the identities \( w = q \) and \( w_t = -w_x \) in the first order terms of the difference between (178) and (179) to obtain a more symmetric relation of the form:

\[
w_t - q_t + \frac{\alpha}{2} q q_t - \frac{\beta}{3} q_{xxx} = w_x - q_x + \frac{\alpha}{2} q q_x - \frac{\beta}{3} q_{xxx},
\]

where we replaced the terms of quadratic order \( \mathcal{O}_2 \) by 0. This equation is of the form

\[
\frac{\partial}{\partial t} G(x, t) = \frac{\partial}{\partial x} G(x, t),
\]

with

\[
G(x, t) = w - q + \frac{\alpha}{4} q^2 - \frac{\beta}{3} q_{xx}.
\]

It has the general solution \( G(x, t) = g(x+t) \). Since the difference \( w - q \) is of linear order \( \mathcal{O}(\alpha, \beta) \) we may write

\[
w = q - \frac{\alpha}{4} q^2 + \frac{\beta}{3} q_{xx} + \alpha g_1(x+t) + \beta g_2(x+t).
\]

According to assumption 7 the disturbance \( \alpha g_1(x+t) + \beta g_2(x+t) \) which is travelling to the left has to vanish. Hence we will restrict ourselves to the subclass of solutions of (188) with \( g_1 = g_2 = 0 \) for physical reasons. Mathematically it is not necessary that \( g_1 = g_2 = 0 \) should hold. Inserting \( w = q - \frac{\alpha}{4} q^2 + \frac{\beta}{3} q_{xx} \) into (178) or (179) and using \( w = q \) and \( w_t = -w_x \) in the first order terms we eventually obtain

\[
qu_t + q_x + \frac{3\alpha}{2} q q_x + \frac{\beta}{3} q_{xxx} = 0.
\]

This is a modified form of the KdV equation due to the term \( q_x \). Mathematically, this term would disappear after a suitable transformation \( q \rightarrow q + C \). Unfortunately in our case this transformation is unphysical since \( C = \mathcal{O}(\alpha^{-1}) \) would be large and the transformation would violate the boundary conditions. Nevertheless, the modified KdV equation (192) admits 1-soliton solutions of the hyperbolic secans form we considered in section 6.1.

6 Exact solution methods for non-linear wave equations

We have encountered special 1-soliton solutions for the CSE (section 4.4) and the KdV equation (section 5.1). Further we have studied the interaction of 2 solitons numerically and analytically. More complicated solutions describing the interaction of \( N \) solitons can also be obtained for a class of nonlinear wave equations including the CSE and the KdV equation. The corresponding technique is called the “inverse scattering transform”
(IST) and was first developed in the papers of Gardner, Greene, Kruskal, Miura [6] for the KdV equation, later generalized by Lax [7] and modified to the CSE by Zakharov and Shabat [8].

Since this theory is mathematically rather sophisticated and beyond the scope of the present course, we will only explain the basic ideas of the IST without going into details. Also we will only consider the case of the KdV equation since the other cases are more involved but similar in spirit.

### 6.1 An introduction into the inverse scattering transform (IST)

In order to motivate the IST technique let us first consider a linear wave equation of the form

$$i q_t = \omega(-i\partial_x)q,$$  \hspace{1cm} (193)

where $\omega$ shall denote a rather arbitrary function to be applied to the “momentum operator” $-i\partial_x$. For example, the linear free 1-dimensional Schrödinger equation is obtained with $\omega(p) = \frac{1}{2m}p^2$ if $\hbar = 1$. The common technique to solve this kind of wave equation is the Fourier transform, comprised by the following equations

\begin{align*}
q(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} b(k, t)e^{ikx} dk, \quad (194) \\
\omega(-i\partial_x)q &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} b(k, t)\omega(k)e^{ikx} dk, \quad (195) \\
ib_t &= \omega(k)b, \quad (196) \\
b(k, t) &= e^{-i\omega(k)t}b(k, 0). \quad (197)
\end{align*}

If the initial value $q(x, 0)$ of the wave function is given, its Fourier transform $b(k, 0)$ has to be multiplied by $e^{-i\omega(k)t}$ and Fourier re-transformed to give the desired solution $q(x, t)$. This procedure is summarized in table 2.

<table>
<thead>
<tr>
<th>$q(x, 0)$</th>
<th>Fourier transform</th>
<th>$b(k, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q(x, t)$</td>
<td>Inverse Fourier transform</td>
<td>$b(k, t)$</td>
</tr>
</tbody>
</table>

Table 2: Solution scheme for linear wave equations of the form (193).

Recall that the Fourier transform of an 1-dimensional object $q(x)$ may be realized as the interference pattern at a large distance if the object is illuminated with coherent monochromatic light. In this sense the function $b(k)$ can be viewed to comprise the
“scattering data” of the object \( q(x) \) and it is possible to recover the object \( q(x) \) from its scattering data \( b(k) \), or to “see” the object. (In 3 dimensions this is the problem of computer tomography.) If the object \( q(x) \) is exposed to a particle beam, we obtain another kind of scattering data by solving the corresponding Schrödinger equation

\[
\psi_{xx} + q \psi = -k^2 \psi. \tag{198}
\]

This is the linear Schrödinger equation for 1-dimensional particles hitting a localized potential \( q(x) \) and has to be distinguished from the non-linear Schrödinger or KdV equations which we want to solve. In the context of the scattering problem we look for eigenfunctions \( \phi, \psi_1, \psi_2 \), i. e. solutions of (198), which additionally satisfy

\[
\phi \sim e^{ikx} \text{ for } x \to -\infty \tag{199}
\]

and hence represent incoming plane waves and

\[
\begin{align*}
\psi_1 & \sim e^{ikx} \\
\psi_2 & \sim e^{-ikx}
\end{align*} \text{ for } x \to \infty \tag{200}
\]

and represent outgoing, transmitted or reflected, waves. If the condition (199) uniquely characterizes the eigenfunction, we may write

\[
\phi = a(k)\psi_1 + b(k)\psi_2, \tag{201}
\]

with generally complex functions \( a(k) \) and \( b(k) \). The quotient

\[
\rho(k) \equiv \frac{b(k)}{a(k)} \tag{202}
\]

is called the “reflection coefficient” and will be considered as the first part of the scattering data. The second part consists of data of the bound states \( \phi_n(x) \) of the potential \( q(x) \) with eigenvalues \( -\kappa_n^2, n = 1, \ldots, N \). Since the potential is localized they exponentially decay for large values of \( x \):

\[
\phi_n(x) \sim C_n e^{-\kappa_n x} \text{ for } x \to \infty. \tag{203}
\]

The reflection coefficient \( \rho(k) \) together with the \( C_n, \kappa_n \) define the “scattering data”, in symbols

\[
\mathcal{S}(q) = \{ \rho(k), (C_n, \kappa_n)_{n=1, \ldots, N} \}. \tag{204}
\]

\( \mathcal{S}(q) \) can also be considered as the “scattering transform” of \( q \), in analogy to the Fourier transform considered above. The scattering transform is designed to solve non-linear wave equations analogously to the solution of linear wave transformations by the Fourier transform. But there are two problems left:

- How do the scattering data depend on \( t \), if the potential \( q \) changes with some parameter \( t \) ?
• How can the scattering transform be inverted, i.e. how can the potential $q$ be reconstructed from the scattering data? (IST)

Concerning the first problem, we should stress that the parameter $t$ has nothing to do with the time of the scattering process. Without any proof we state only the solution of this problem: If the potential $q(x,t)$ changes according to the KdV equation, then the scattering data have the following, very simple $t$-dependence:

$$\rho(k,t) = \rho(k,0)e^{ik^3t}, \quad (205)$$
$$C_n(t) = C_n(0)e^{ik^3t}, \quad (206)$$
$$\kappa_n(t) = \kappa_n(0). \quad (207)$$

Concerning the second problem of the IST, the solution was known before it was applied to the $N$ soliton problem. In 3-dimensional scattering theory of spherically symmetric potentials it is well-known that the scattering amplitude is proportional to the Fourier transform of the potential, see e.g. [9], chapter 8, (8.129). For 1-dimensional scattering the recipe reads as follows: Define

$$F(x) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \rho(k)e^{ikx}dk - \sum_{n=1}^{N} C_n e^{-\kappa_n x} \quad (208)$$

and solve the so-called Gelfand-Levitan-Marchenko integral equation

$$K(x,y) + F(x+y) + \int_{x}^{\infty} K(x,z)F(z+y) = 0. \quad (209)$$

Then the potential is obtained by

$$q(x) = -2\frac{d}{dx}K(x,x). \quad (210)$$

Since there is no general method of solving (209), the IST has only be applied to the $N$-soliton problem in the special case of “reflectionless potentials”, i.e. where $\rho(k) = 0$ for all $k$. We will sketch the solution for this case. Using

$$F(x+y) = -i \sum_{n} C_n e^{-\kappa_n(x+y)} \equiv \sum_{n} g_n(x)h_n(y), \quad (211)$$

and the ansatz

$$K(x+y) = \sum_{n} w_n(x)h_n(y), \quad (212)$$

the Gelfand-Levitan-Marchenko equation (209) reduces to

$$\sum_{n} w_n(x)h_n(y) + \sum_{n} g_n(x)h_n(y) + \int_{x}^{\infty} \sum_{nm} w_n(x)h_n(z)g_m(z)h_m(y)dz = 0. \quad (213)$$
Since the functions \( h_n, n = 1, \ldots, N \) are linearly independent, this implies
\[
w_m(x) + g_m(x) + \sum_n w_n(x) \int_x^\infty g_m(z)h_n(z)\,dz = 0. \tag{214}
\]
In vector notation this reads
\[
P(x)\vec{w}(x) = -\vec{g}(x), \tag{215}
\]
where
\[
P_{mn}(x) \equiv \delta_{mn} + \int_x^\infty g_m(z)h_n(z)\,dz. \tag{216}
\]
(215) has the formal solution
\[
\vec{w}(x) = -P^{-1}(x)\vec{g}(x). \tag{217}
\]
Hence
\[
K(x) = \sum_n w_n(x)h_n(x) = \vec{h}(x) \cdot \vec{w}(x) \tag{218}
\]
\[
= -\vec{h}(x) \cdot P^{-1}(x)\vec{g}(x) = -\sum_{mn} h_n(x)P_{mn}^{-1}(x)g_m(x) \tag{219}
\]
\[
= \sum_{mn} P_{mn}^{-1}(x) \frac{d}{dx} P_{mn}(x) \tag{220}
\]
\[
= \text{Tr} \left( P^{-1} \frac{d}{dx} P \right) \tag{221}
\]
\[
= \frac{1}{\det P} \frac{d}{dx} \det P = \frac{d}{dx} (\log \det P). \tag{222}
\]
Finally, by (210)
\[
q(x) = -2\frac{d^2}{dx^2} \log \det P. \tag{223}
\]
The scheme for the IST solution of the \( N \) soliton problem is contained in the following table 3.

References


Table 3: Solution scheme for nonlinear wave equations.

\[ u(x, 0) \xrightarrow{\text{Direct scattering}} S(q, 0) = (\{\kappa_n, c_n(0)\}_{n=1}^{N}, \rho(k, 0)) \]
\[ \downarrow \]
\[ u(x, t) \xleftarrow{\text{Inverse scattering}} S(q, t) = (\{\kappa_n, c_n(t)\}_{n=1}^{N}, \rho(k, t)) \]


7 Appendices

7.1 Appendix 1
7.2 Appendix 2
7.3 Appendix 3
7.4 Appendix 4a
7.5 Appendix 4b
7.6 Appendix 5
Appendix 1
3.3.1: Small damping, no driving force

Harmonic oscillator with friction

\[ k = 1; \ \Gamma = 0.01; \]
\[ \text{sol} = \text{NDSolve}[\{\varphi'[t] = \omega[t], \ \omega'[t] = -k\varphi[t] - 2\Gamma\omega[t], \ \varphi[0] = 1, \ \omega[0] = 0\}, \]
\[ \{\varphi[t], \ \omega[t]\}, \{t, 0, 60\pi\}]
\[
\{(\varphi[t] \rightarrow \text{InterpolatingFunction}[\{\{0., 188.496\}\}, <>][t],
\omega[t] \rightarrow \text{InterpolatingFunction}[\{\{0., 188.496\}\}, <>][t]\}]
\]
\[ \text{ParametricPlot[Evaluate[\{\varphi[t], \ \omega[t]\} //. \text{sol}],}
\]
\[ \{t, 0, 60\pi\}, \text{PlotStyle} \rightarrow \text{Hue[0]}, \text{PlotRange} \rightarrow \text{All}, \text{PlotPoints} \rightarrow 100]\]

- Damped motion in phase space
The swept phase space area $I(E)$ as a function of time

$$\text{listI} = \text{Table}[\{2\pi (n-1) + \pi, \text{NIntegrate}[\text{Evaluate}[\omega[t]^2 \text{/. sol}], \{t, 2\pi (n-1), 2\pi n\}]\}, \{n, 1, 30\}]$$

$$\{(\pi, 2.95229), (3\pi, 2.6035), (5\pi, 2.29599), (7\pi, 2.02483), (9\pi, 1.78569), (11\pi, 1.5748), (13\pi, 1.38881), (15\pi, 1.22479), (17\pi, 1.08014), (19\pi, 0.95257), (21\pi, 0.840066), (23\pi, 0.74085), (25\pi, 0.653351), (27\pi, 0.576187), (29\pi, 0.508136), (31\pi, 0.448123), (33\pi, 0.395197), (35\pi, 0.348522), (37\pi, 0.30736), (39\pi, 0.271059), (41\pi, 0.239045), (43\pi, 0.210813), (45\pi, 0.185914), (47\pi, 0.163957), (49\pi, 0.144592), (51\pi, 0.127514), (53\pi, 0.112454), (55\pi, 0.0991719), (57\pi, 0.0874586), (59\pi, 0.0771288)\}$$
Comparison with the theoretical result

\[ FP = \text{Plot}[\pi \text{Exp}[-2 \pi t], \{t, 0, 60 \pi\}, \text{PlotStyle} \to \text{Hue}[0.75]] \]
Pendulum with friction

- Moderate Amplitude

\[ k = 1; \Gamma = 0.01; \]
\[ \text{sol = NDSolve}\{\varphi'[t] = \omega[t], \omega'[t] = -k \sin[\varphi[t]] - 2 \Gamma \omega[t], \varphi[0] = 1, \omega[0] = 0\}, \}
\{\varphi[t], \omega[t]\}, \{t, 0, 60 \pi\}\]

\[ \{\varphi[t] \to \text{InterpolatingFunction}[[0., 188.496]], <>[t], \]
\[ \omega[t] \to \text{InterpolatingFunction}[[0., 188.496]], <>[t]\}\} \]
- Damped motion in phase space

```math
\text{ParametricPlot[Evaluate[\{\varphi[t], \omega[t]\} /. sol],}
\{t, 0, 60\pi\}, \text{PlotStyle} \to \text{Hue}[0], \text{PlotRange} \to \text{All, PlotPoints} \to 100]}
```
Dissipation of energy

\[ \text{Plot}\left[\text{Evaluate}\left[k \left(1 - \cos(\varphi(t))\right) + \frac{1}{2} \omega(t)^2 \right], \{t, 0, 60\pi\}, \text{PlotStyle} \rightarrow \text{Hue}[0.4]\right] \]
The phase space area $I(E)$ as a function of time

```math
listI = {}; For[n = 1; oldroot = 0;
    newroot = FindRoot[(\[omega][t] //. sol)[[1]] == 0, {t, 2 \[Pi]}][[1, 2]],
    n <= 25, n++, \[phi]1 = \[omega][t] //. sol //. t -> oldroot;
    oldroot + newroot
    listI = Append[listI, \[phi]1];
    oldroot = newroot;
    nroot = FindRoot[(\[omega][t] //. sol)[[1]] == 0, {t, oldroot + 2 \[Pi]}][[1, 2]]; listI

{\{3.33554, 2.78926\}, \{9.9014, 2.44218\}, \{16.5759, 2.14038\}, \{23.1301, 1.87743\},
 {29.6498, 1.64795\}, \{36.1372, 1.4474\}, \{42.5994, 1.27192\}, \{49.039, 1.11822\},
 {55.459, 0.983491\}, \{61.862, 0.865258\}, \{68.2503, 0.761477\}, \{74.6256, 0.670315\},
 {80.9898, 0.590196\}, \{87.3441, 0.519747\}, \{93.6898, 0.457769\},
 {100.028, 0.403238\}, \{106.36, 0.355251\}, \{112.685, 0.313025\},
 {119.006, 0.275851\}, \{125.322, 0.243114\}, \{131.635, 0.214278\},
 {137.944, 0.188873\}, \{144.249, 0.166491\}, \{150.553, 0.146777\}, \{156.853, 0.12939\}}

LPP = ListPlot[listI, PlotStyle -> Hue[0]]
```

- Graphics -
- Comparison with the theoretical result for $\Gamma << 1$:

$$FPP = \text{Plot}[2.7892585884622143*\text{Exp}[-2\Gamma (t - 6.671075605173357/2)], \{t, 0, 60\pi\}, \text{PlotStyle} \rightarrow \text{Hue}[0.4]]$$

- Graphics -

Show[LPP, FPP]
- Comparison with harmonic oscillator

```math
Show[LPP, FPP, LP, FP]
```

- Graphics -

- Large Amplitude

```math
k = 1; \Gamma = 0.01;
sol = NDSolve[{\phi'[t] = \omega[t], \omega'[t] = -k \sin[\phi[t]] - 2 \Gamma \omega[t], \phi[0] = 3.1, \omega[0] = 0},
{\phi[t], \omega[t]}, \{t, 0, 46\pi}]}
```

```math
{\phi[t] \rightarrow \text{InterpolatingFunction}[\{\{0., 144.513\}\}, <>][t],
\omega[t] \rightarrow \text{InterpolatingFunction}[\{\{0., 144.513\}\}, <>][t]}
- Damped motion in phase space

\[
\text{ParametricPlot[Evaluate[\{\varphi[t], \omega[t]\} /. \text{sol}],}
\{t, 0, 46 \, \pi\}, \text{PlotStyle} \rightarrow \text{Hue[0]}, \text{PlotRange} \rightarrow \text{All}, \text{PlotPoints} \rightarrow 100]
\]

- Graphics -

\[
\text{ParametricPlot[Evaluate[\{\varphi[t], \omega[t]\} /. \text{sol}],}
\{t, 0, 13\}, \text{PlotStyle} \rightarrow \text{Hue[0]}, \text{PlotRange} \rightarrow \text{All}]
\]

- Graphics -

- The phase space area I(E), swept within one period, as a function of time

\[
\varphi_1 = ((\varphi[t] /. \text{sol}) /. t \rightarrow 0)[[1]]
\]

3.1


\[ e = \left( k (1 - \cos(\psi[t])) + \frac{1}{2} \omega[t]^2 \right) \text{ sol} \] / . \ t \to 0 \)[[1]]

1.99914

\[ \text{NIntegrate}\left[\sqrt{2 (e - k (1 - \cos(\phi))}, \{\phi, -\phi1, \phi1}\right] \]

7.99004

\text{listI} = \{\}; \text{For} [n = 1; \text{oldroot} = 0;
\text{newroot} = \text{FindRoot}[\omega[t] \text{ sol}][[1]] = 0, \{t, 8}\][[1, 2]]; \text{droot} = 5,
\text{n} \leq 14, \text{n}++;
\text{e} = \left( k (1 - \cos(\psi[t])) + \frac{1}{2} \omega[t]^2 \right) \text{ sol} \] / . \ t \to \text{oldroot} \)[[1]];
\text{\varphi1} = ((\omega[t]) \) \text{ sol} \] / . \ t \to \text{oldroot} \)[[1]];
\text{i} = \text{NIntegrate}\left[\sqrt{2 (e - k (1 - \cos(\phi))}, \{\phi, -Abs[\psi1], Abs[\psi1]\}\right];
\text{listI} = \text{Append}[\text{listI}, \{\text{oldroot}, \text{i}\}\]; \text{oldroot} = \text{newroot};
\text{newroot} = \text{FindRoot}[\omega[t] \text{ sol}][[1]] = 0, \{t, \text{oldroot} + \text{droot}\][[1, 2]];
\text{droot} = \text{newroot} - \text{oldroot}; \text{listI}

\{[0, 7.99004], [7.99731, 7.03508], [13.1254, 6.35404], [17.7911, 5.78937],
[22.1846, 5.30303], [26.3893, 4.87566], [30.4521, 4.49537],
[34.4032, 4.15399], [38.2633, 3.84546], [42.0477, 3.56521],
[45.768, 3.30961], [49.4331, 3.0757], [53.0503, 2.86107], [56.6255, 2.66364]}

\text{LPL} = \text{ListPlot}[\text{listI}, \text{PlotStyle} \to \text{Hue}[0]]
Comparison with the theoretical result for $\Gamma << 1$:

\[
FPL = \text{Plot}\left[\{4.495371795489376 \times \text{Exp}\left[-2 \Gamma (t - 30.452110072692125)\right]\},
\{t, 0, 60\}, \text{PlotStyle} \rightarrow \text{Hue}[0.4]\right];
\]

Show[LPL, FPL]
Appendix 2: Nonlinear Oscillations and chaotic motion

Linear oscillator with periodic driving force

\[ k = 0.1; \, \Gamma = 0.2; \, \lambda = 1; \]
\[ \text{sol = NDSolve[}\{\phi'[t] = \omega[t], \, \omega'[t] = -k\phi[t] - 2 \Gamma \omega[t] + \lambda \cos[t], \, \phi[0] = 0, \, \omega[0] = 0\}, \]
\[ \{\phi[t], \omega[t]\}, \{t, 0, 20\pi\}] \]
\[ {{\phi[t}] \to \text{InterpolatingFunction}[\{\{0., 62.8319\}\}, \&](t),} \]
\[ {{\omega[t] \to \text{InterpolatingFunction}[\{\{0., 62.8319\}\}, \&](t)}} \]

\text{ParametricPlot[Evaluate[}\{\phi[t], \omega[t]\}//.\text{sol}, \{t, 0, 20\pi\}, \text{PlotStyle} \to \text{Hue}[0]]}
Pendulum with periodic driving force

\[ k = 0.1; \Omega = 1; \Gamma = 0.2; \lambda = 1; \]
\[ EQ = \{ \dot{\varphi}[t] = \omega[t], \omega'[t] = -k \sin(\varphi[t]) - 2 \Gamma \omega[t] + \lambda \cos(\Omega t) \}; \]
\[ For[i = 1; \; \; poin = \{ \}; \; \; IV = \{ \varphi[0] = 0, \omega[0] = 0 \}, \; \; 1 \leq i, \; i++; \]
\[ sol = NDSolve[Join[EQ, IV], \{ \varphi[t], \omega[t] \}, \{ t, 0, 16\pi/\Omega \}]; \]
\[ If[i > 1, poin = Join[poin, Table[ \]
\[ Evaluate[(\Mod[\varphi[t] + \pi, 2\pi] - \pi, \omega[t])/./sol[[1]] //. t \to n 2\pi/\Omega], \{ n, 1, 8 \}]]; \]
\[ iv = Evaluate[(\Mod[\varphi[t] + \pi, 2\pi] - \pi, \omega[t])/./sol[[1]] //. t \to 16\pi/\Omega]; \]
\[ IV = \{ \varphi[0] = iv[[1]], \omega[0] = iv[[2]] \}; \]
\[ ParametricPlot[Evaluate[(\Mod[\varphi[t] + \pi, 2\pi] - \pi, \omega[t])/./sol], \]
\[ \{ t, 0, 16\pi/\Omega \}, \text{PlotStyle} \to \text{Hue}[0]]; \]
\[ ListPlot[poin, \text{PlotRange} \to \{-1, 1\}, \{-1, 1\}] \]
Poincaré section

Pendulum with periodic driving force:
Forced oscillation with large amplitude

\[ \Omega = 0.9; \Gamma = 0.1; \lambda = 1; \text{EQ} = \{ \varphi'[t] = \omega[t], \omega'[t] = -\sin[\varphi[t]] - 2 \Gamma \omega[t] + \lambda \cos[\Omega t] \} \]

\[ \text{For}[i = 1; \text{poin} = \{}; \text{IV} = \{ \varphi[0] = 0, \omega[0] = 0.242124 \}, \text{i} \leq 6, \]

\[ \text{i}++, \text{sol} = \text{NDSolve}[	ext{Join}[\text{EQ}, \text{IV}], \{ \varphi[t], \omega[t] \}, \{ t, 0, 16 \pi / \Omega \}]; \]

\[ \text{If}[\text{i} > 3, \text{poin} = \text{Join}[\text{poin}, \text{Table[}} \]

\[ \text{Evaluate}[\{ \text{Mod}[\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] \}] //. \text{sol}[[1]] //. t \rightarrow n 2 \pi / \Omega, \{ n, 1, 8 \}]]; \]

\[ \text{iv} = \text{Evaluate}[\{ \text{Mod}[\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] \}] //. \text{sol}[[1]] //. t \rightarrow 16 \pi / \Omega; \]

\[ \text{IV} = \{ \varphi[0] = \text{iv}[[1]], \omega[0] = \text{iv}[[2]] \}; \]

\[ \text{ParametricPlot}[\text{Evaluate}[\{ \text{Mod}[\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] \}] //. \text{sol}, \]

\[ \{ t, 0, 16 \pi / \Omega \}, \text{PlotStyle} \rightarrow \text{Hue}[0] \}; \]

\[ \text{ListPlot}[\text{poin}, \text{PlotRange} \rightarrow \{-\pi, \pi\}, \{-4, 4\}] \]
Poincaré section
Pendulum with periodic driving force:
Forced oscillation with 2 loopings and triple period

\[ \Omega = 0.9; \Gamma = 0.1; \lambda = 1; EQ = \{ \varphi'[t] = \omega[t], \omega'[t] = -\sin[\varphi[t]] - 2 \Gamma \omega[t] + \lambda \cos[\Omega t] \}; \]
For\[i = 1; \text{poin} = {}; \text{IV} = \{ \varphi[0] = 0, \omega[0] = 0.242125 \}, i \leq 6, \]
i++, \text{sol} = \text{NDSolve}[\text{Join}[EQ, \text{IV}], \{ \varphi[t], \omega[t] \}, \{ t, 0, 16 \pi / \Omega \}];
If\[i > 3, \text{poin} = \text{Join}[	ext{poin}, \text{Table}[
\text{Evaluate}[\text{Mod} [\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] / \text{.} \text{sol}[[1]] / \text{.} t \to n 2 \pi / \Omega, \{ n, 1, 8 \}]]; \]
iv = \text{Evaluate}[\text{Mod} [\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] / \text{.} \text{sol}[[1]] / \text{.} t \to 16 \pi / \Omega];
IV = \{ \varphi[0] = \text{iv}[[1]], \omega[0] = \text{iv}[[2]] \};
\text{ParametricPlot}[\text{Evaluate}[\{ \text{Mod} [\varphi[t] + \pi, 2 \pi] - \pi, \omega[t] \} / \text{.} \text{sol}],[t, 0, 16 \pi / \Omega], \text{PlotStyle} \to \text{Hue}[0]]; \]
\text{ListPlot}\{\text{poin}, \text{PlotRange} \to \{ -\pi, \pi \}, \{-4, 4 \}] \]
Poincaré section

Chaotic oscillation and strange attractor

Ω = 0.5; Γ = 0.1; λ = 1.3; 
\[ EQ = \{ \dot{\psi} = \omega[t], \ \dot{\omega} = -\sin[\psi[t]] - 2 \Gamma \omega[t] + \lambda \cos[\Omega t] \}; \]
Nperiod = 6; 
For[i = 1; poin = {}; iv = \{ \psi[0] = 0, \omega[0] = 0 \}, i \leq 10, 
i++, 
sol = NDSolve[Join[EQ, IV], \{ \psi[t], \omega[t] \}, \{ t, 0, Nperiod 2\pi / \Omega \}]; 
If[i > 4, poin = Join[poin, Table[Evaluate[ 
    \{ Mod[\psi[t] + \pi, 2\pi] - \pi, \omega[t] \} //. \ sol[[1]] //. t \rightarrow n 2 \pi / \Omega, \{ n, 1, Nperiod \}] ], 
iv = Evaluate[\{ Mod[\psi[t] + \pi, 2\pi] - \pi, \omega[t] \} //. \ sol[[1]] //. t \rightarrow Nperiod 2\pi / \Omega]; 
IV = \{ \psi[0] = iv[[1]], \omega[0] = iv[[2]] \}; 
ParametricPlot[Evaluate[\{ Mod[\psi[t] + \pi, 2\pi] - \pi, \omega[t] \} //. \ sol], 
\{ t, 0, Nperiod 2\pi / \Omega \}, PlotStyle \rightarrow Hue[0]]; 
ListPlot[poin, PlotRange \rightarrow \{ \{-\pi, \pi\}, \{-4, 4\} \}];

- Graphics -

Appendix02.nb
Poincaré section

[Graph showing a Poincaré section with about 60,000 points]

Poincaré section with about 60,000 points:
Strange attractor

\[ \dot{\varphi} = \omega, \quad \dot{\omega} = -\sin(\varphi) - 2 \Gamma \omega + \lambda \cos(\Omega t) \]

\( \Omega = 0.5; \Gamma = 0.1; \lambda = 1.3; \) EQ = \{\varphi'[t] = \omega[t], \quad \omega'[t] = -\sin(\varphi[t]) - 2 \Gamma \omega[t] + \lambda \cos(\Omega t)\};

Nperiod = 3; For[i = 1; poin = {}; IV = \{\varphi[0] = 0, \omega[0] = 0\}, i \leq 20000,

i++, sol = NDSolve[Join[EQ, IV], \{\varphi[t], \omega[t]\}, \{t, 0, Nperiod 2 \pi / \Omega\}];

If[i > 10, poin = Join[poin, Table[Evaluate[

{Mod[\varphi[t] + \pi, 2 \pi] - \pi, \omega[t]} / sol[[1]] // t \rightarrow n 2 \pi / \Omega, \{n, 1, Nperiod\}]]];

iv = Evaluate[{Mod[\varphi[t] + \pi, 2 \pi] - \pi, \omega[t]} // sol[[1]] // t \rightarrow Nperiod 2 \pi / \Omega];

IV = \{\varphi[0] = iv[[1]], \omega[0] = iv[[2]]\}]

Appendix02.nb
Fractal dimension of the strange attractor

dimlist = {}; For[n = 0, n <= 10, n++, \(\varepsilon = \text{Exp}[-n]\); poinround = \(\text{Union}[\varepsilon \cdot \text{Round}[\text{poin}/\varepsilon]]\); Nround = \(\text{Length}[\text{poinround}]\); dimlist = \(\text{Append}[\text{dimlist}, \{n, N[\text{Log}[\text{Nround}]]\}]\); \(\text{Show}[\text{Graphics}[\text{Table}[\text{Rectangle}[\text{poinround}[[i, 1]] - \varepsilon/2, \text{poinround}[[i, 2]] - \varepsilon/2], \{\text{poinround}[[i, 1]] + \varepsilon/2, \text{poinround}[[i, 2]] + \varepsilon/2\}], \{i, \text{Length}[\text{poinround}]]], \text{PlotRange} \to \{-4, 4\}, \{-4, 4\}]\]
Log of # of pixels versus negative Log of pixel size

redlist = Table[dimlist[[i]], {i, 1, 5}]

{{0, 3.17805}, {1, 4.60517}, {2, 5.98645}, {3, 7.46851}, {4, 8.77028}}

f[x_] = Fit[redlist, {1, x}, x]

3.19213 + 1.40478 x

3.1921340098412108^ + 1.4047803063009157^-x
⇒ Dimension of (Poincaré section of)
strange attractor ≈ 1.4
Appendix 3

3.3.3.1: Bistable response of the Duffing oscillator

We demonstrate two different forced periodic motions of the Duffing oscillator with two different initial conditions. All other parameters remain the same.

\[ \omega_0 = \sqrt{0.1}; \omega = 1.785; \Gamma = 0.2; a = 5/2; \lambda = 0.2; \]
\[ \text{EQ} = \{ \phi'[t] = v[t], v'[t] = -\omega^2 \phi[t] - \lambda \phi[t] - 3 - 2 \Gamma v[t] + 2 a \cos(\omega t) \}; \]
\[ \text{For}[i = 1, IV = \{ \phi[0] = 0, v[0] = 1 \}, i < 3, i++]; \]
\[ \text{soll} = \text{NDSolve}[\text{Join}[\text{EQ}, \text{IV}], \{ \phi[t], v[t] \}, \{ t, 0, 16\pi/\omega \}]; \]
\[ \text{iv} = \text{Evaluate}[\{ \phi[t], v[t] \} /. \text{soll}[[1]] /. t \rightarrow 16\pi/\omega]; \]
\[ \text{IV} = \{ \phi[0] = \text{iv}[[1]], v[0] = \text{iv}[[2]] \}; \]
\[ \text{PP1} = \text{ParametricPlot}[\text{Evaluate}[\{ \phi[t], v[t] \} /. \text{soll}], \{ t, 0, 16\pi/\omega \}, \text{PlotStyle} \rightarrow 0 \text{Hue[0]}]] \]
\( \omega_0 = \sqrt{0.1} \); \( \omega = 1.785 \); \( \Gamma = 0.2 \); \( a = \frac{5}{2} \); \( \lambda = 0.2 \);

\[ \text{EQ} = \{ \phi'[t] = v[t], \ v'[t] = -\omega^2 \phi[t] - \lambda \phi[t]^{\gamma} - 2 \Gamma v[t] + 2a \cos[\omega t]\}; \]

For \( i = 1; \ IV = \{ \phi[0] = 0, \ v[0] = -1 \}, \ i \leq 3, \ i++; \)

\( \text{sol} = \text{NDSolve}[[\text{Join}[\text{EQ}, \ IV], \{ \phi[t], \ v[t] \}, \{ t, \ 0, \ 16\pi/\omega \}]]; \)

\( \text{iv} = \text{Evaluate}[[\{ \phi[t], \ v[t] \} //. \ \text{sol}[i]] //. \ t \to 16\pi/\omega]; \)

\( \text{IV} = \{ \phi[0] = \text{iv}[[1]], \ v[0] = \text{iv}[[2]]\}; \)

\( \text{PP2} = \text{ParametricPlot}[
\quad \text{Evaluate}[[\{ \phi[t], \ v[t] \} //. \ \text{sol}, \{ t, \ 0, \ 16\pi/\omega \}, \text{PlotStyle} \to \text{Hue}[0.7]]]
\)
Show[PP1, PP2]
- We see that one oscillation has a large amplitude, the other a small one.

- Next we vary the frequency $\omega$ of the driving force and choose random initial values. We plot the amplitude of the forced oscillation versus $\omega$.

\[
\text{AList} = \{}; \omega 0 = \sqrt{0.1}; \Gamma = 0.2; a = 5/2; \lambda = 0.2; \text{For}[\omega = 0.9, \omega < 5, \omega = \omega + 0.05, \\
\text{EQ} = \{\phi'[t] = \nu[t], \nu'[t] = -\omega^2 \phi[t] - \lambda \phi[t] \nu[t] + 2 \Gamma \nu[t] + 2 a \cos[\omega t]\}; \\
\text{Do}[\text{For}[i = 1; \text{IV} = \{\phi[0] = \pi (2 \text{Random[]}) - 1, \nu[0] = 10 (2 \text{Random[]}) - 1\}; \\
\text{i} \leq 16, \text{i}++, \text{sol} = \text{NDSolve}[\text{Join}[\text{EQ}, \text{IV}], \{\phi[t], \nu[t]\}, \{t, 0, 4 \pi / \omega\}] ; \\
\text{iv} = \text{Evaluate}[[\phi[t], \nu[t]] / . \text{sol}[1] / . t \rightarrow 4 \pi / \omega]; \\
\text{IV} = \{\phi[0] = \text{iv}[1], \nu[0] = \text{iv}[2]\}] ; \\
\text{sol} = \text{NDSolve}[\text{Join}[\text{EQ}, \text{IV}], \{\phi[t], \nu[t]\}, \{t, 0, 4 \pi / \omega\}] ; \\
\text{t} = \text{FindRoot}[\text{Evaluate}[\nu[t] / . \text{sol}[[1]]] = 0, \{t, 1\}] ; \\
\text{A} = \text{Abs}[\text{Evaluate}[\phi[t] / . \text{sol}[[1]] / . \text{solt}]; \text{AList} = \text{Append}[\text{AList}, \{\omega, \text{A}\}, \{10\}] ; \\
\text{AP} = \text{ListPlot}[\text{AList}, \text{PlotRange} \rightarrow \{0, 6\}] 
\]
We see that around $\omega = 2$ there is a region where bi-stable response occurs.

There exists an approximate theory of bi-stable response by Duffing. According to this theory the resonance curve of a Duffing oscillator is given by $f(A, \omega) = 4 \Gamma^2 A^2 \omega^2 + \left((\omega_0^2 - \omega^2) A + \frac{3}{4} \lambda A^3\right)^2 = \text{const.}$ We will compare our numerical results with Duffing's theory:

```
<< Graphics`ImplicitPlot`

Clear[\[Omega], A]; IP = ImplicitPlot[4 \[Gamma]^2 A^2 \[Omega]^2 + \left((\omega_0^2 - \omega^2) A + 3/4 \lambda A^3\right)^2 = 25, 
{\[Omega], 0, 5}, {A, 0, 6}, PlotPoints -> 250, PlotStyle -> Hue[0]]
```
We see that the oscillation with small amplitude is well represented by Duffing’s resonance curve. The large oscillation deviates from Duffing’s theory. The deviation increases with smaller $\omega$. This seems reasonable since for small driving frequencies $\omega$ the Duffing oscillator describes a different motion: It tries to follow a slowly varying time-dependent potential. Nevertheless, Duffing’s equation is qualitatively correct in the bi-stable regime.
Appendix 4a

3.3.3.2: Perturbation series solution of the Duffing oscillator up to $O(\lambda^2)$-terms

$$\dot{x} + 2\gamma \dot{x} + \alpha x + \lambda x^3 = A p e^{i\omega t} + CC$$

- nax is the maximal power of the nonlinear coefficient $\lambda$ to be considered in the perturbation series

$$x(t) = \sum_{n=0}^{\text{nax}} \sum_{m=-2n+1}^{2n+1} x[n,m] \lambda^n p^m e^{i\omega t}$$

nax = 2; X = Flatten[Array[x, {nax + 1, 4 nax + 3}, {0, -2 nax - 1}]]

{x[0, -5], x[0, -4], x[0, -3], x[0, -2], x[0, -1], x[0, 0], x[0, 1], x[0, 2], x[0, 3],
  x[0, 4], x[0, 5], x[1, -5], x[1, -4], x[1, -3], x[1, -2], x[1, -1], x[1, 0],
  x[1, 1], x[1, 2], x[1, 3], x[1, 4], x[1, 5], x[2, -5], x[2, -4], x[2, -3],
  x[2, -2], x[2, -1], x[2, 0], x[2, 1], x[2, 2], x[2, 3], x[2, 4], x[2, 5]}

Y = Select[X, (OddQ[#[[2]]]) && Abs[#[[2]]] <= 2 #[[1]] + 1] &

{x[0, -1], x[0, 1], x[1, -3], x[1, -1], x[1, 1],
  x[1, 3], x[2, -5], x[2, -3], x[2, -1], x[2, 1], x[2, 3], x[2, 5]}

z = Sum[x[n, m] w^m \lambda^n, {n, 0, nax}, {m, -(2 n + 1), 2 n + 1, 2}]

\[
\begin{align*}
\lambda^2 x[2, -5] &+ \lambda^2 x[2, -3] + \lambda^2 x[2, -1] + w \lambda x[1, 1] + w^3 \lambda x[1, 3] + \\
\lambda^2 x[2, -5] &+ \lambda^2 x[2, -3] + \lambda^2 x[2, -1] + w \lambda x[1, 1] + w^3 \lambda x[1, 3] + \\
\lambda^2 x[2, -5] &+ \lambda^2 x[2, -3] + \lambda^2 x[2, -1] + w \lambda x[1, 1] + w^3 \lambda x[1, 3] + \\
\lambda^2 x[2, -5] &+ \lambda^2 x[2, -3] + \lambda^2 x[2, -1] + w \lambda x[1, 1] + w^3 \lambda x[1, 3] + \\
\lambda^2 x[2, -5] &+ \lambda^2 x[2, -3] + \lambda^2 x[2, -1] + w \lambda x[1, 1] + w^3 \lambda x[1, 3]
\end{align*}
\]
\[ A \text{ is the amplitude and } p \text{ the phase of the external periodic force} \]

\[ \text{EQ} = \text{Flatten[} \]
\[ \text{Table}\{x[n, m] = h[m] \text{ Coefficient[Coef} \]
\[ \text{ficient[A p w + A / (p w) - } \lambda x^3, w^m], \lambda, n], \]
\[ \{n, 0, \text{na}x\}, \{m, -(2 n + 1), 2 n + 1, 2\}\}]] \]
\[ \{x[0, -1] == \frac{A h[-\omega]}{p}, x[0, 1] == A p h[\omega], x[1, -3] == -h[-3 \omega] x[0, -1]^3, \]
\[ x[1, -1] == -3 h[-\omega] x[0, -1]^2 x[0, 1], x[1, 1] == -3 h[\omega] x[0, -1] x[0, 1]^2, \]
\[ x[1, 3] == -h[3 \omega] x[0, 1]^3, x[2, -5] == -3 h[-5 \omega] x[0, -1]^2 x[1, -3], \]
\[ x[2, -3] == h[-3 \omega] (-6 x[0, -1] x[0, 1] x[1, -3] - 3 x[0, -1]^2 x[1, -1]), x[2, -1] == \]
\[ h[-\omega] (-3 x[0, 1]^2 x[1, -3] - 6 x[0, -1] x[0, 1] x[1, -1] - 3 x[0, -1]^2 x[1, 1]), \]
\[ x[2, 1] == h[\omega] (-3 x[0, 1]^2 x[1, -1] - 6 x[0, -1] x[0, 1] x[1, 1] - 3 x[0, -1]^2 x[1, 3]), \]
\[ x[2, 3] == h[3 \omega] (-3 x[0, 1]^2 x[1, 1] - 6 x[0, -1] x[0, 1] x[1, 3]), \]
\[ x[2, 5] == -3 h[5 \omega] x[0, 1]^2 x[1, 3]\} \]
\[ \text{sol} = \text{Solve[EQ, } \gamma\]
\[ \{h[1, -5] \rightarrow 3 A^5 x[-5 \omega] h[-3 \omega] h[-\omega]^5, \]
\[ x[2, -3] \rightarrow 3 A^5 h[-3 \omega] h[\omega] h[-\omega]^4 (2 h[-3 \omega] + 3 h[-\omega]), \]
\[ x[1, -1] \rightarrow \frac{3 A^5 h[-\omega] h[\omega] (h[-3 \omega] + 6 h[-\omega] + 3 h[\omega])}{p}, \]
\[ x[2, 1] \rightarrow 3 A^5 p h[-\omega] h[\omega] (3 h[-\omega] + 6 h[\omega] + 3 h[-\omega]), \]
\[ x[2, 3] \rightarrow 3 A^5 p^3 h[-\omega] h[\omega] (3 h[\omega] + 2 h[3 \omega]), \]
\[ x[2, 5] \rightarrow 3 A^5 p^5 h[\omega] (3 h[\omega] h[5 \omega], x[1, -3] \rightarrow \frac{3 A^3 h[-3 \omega] h[-\omega]^3}{p^3}, \]
\[ x[1, -1] \rightarrow \frac{3 A^3 h[-\omega]^3 h[\omega]}{p}, x[1, 1] \rightarrow 3 A^3 p h[-\omega] h[\omega]^3, \]
\[ x[1, 3] \rightarrow -A^3 p^3 h[\omega]^3 h[3 \omega], x[0, 1] \rightarrow A p h[\omega], x[0, -1] \rightarrow \frac{A h[-\omega]}{p}\}\} \]
\[ \text{The spectral response function is } h(\omega) = \frac{1}{a - e^{2 i \gamma \omega}} \]

We insert some numerical values for \( (\alpha, \ \omega, \ \gamma, \ \lambda) \) and calculate the \( e^{i \omega t} \) term of the nonlinear response.

\[ f[A_,] = \text{Simplify[} \text{Sum[} x[n, 1] \lambda^n, \{n, 0, \text{na}x\}\} / (A p) //. \text{sol[1][1]]) //. \]
\[ \{h[\omega] \rightarrow \frac{1}{0.1 - \omega^2 + 0.2 i \omega}, \omega \rightarrow 1, \lambda \rightarrow 0.1\} \]
\[ (-1.05882 - 0.235294 i) (1 + (0.373702 + 0.083045 i) A^2 + (0.416908 + 0.125561 i) A^3) \]
The nonlinear Schrödinger equation would contain the term $f(\psi^2)\psi$ with $f$ given by:

$$\text{ComplexExpand}[2 \text{Re}[f[A]]]$$

$$-2.11765 - 0.75229 A^2 - 0.823776 A^4$$

Plot[Re[f[A]], {A, 0, 1}]
Appendix 4b

3.3.3.2: Perturbation series solution of the Duffing oscillator up to $O(\lambda^5)$-terms

\[ \ddot{x} + 2\gamma \dot{x} + \alpha x + \lambda x^3 = Ap e^{i\omega t} + CC \]

- **nax** is the maximal power of the nonlinear coefficient $\lambda$ to be considered in the perturbation series

\[ x(t) = \sum_{n=0}^{nax} \sum_{m=-2n-1}^{2n+1} x[n, m] \lambda^n e^{i\omega t} \]

We will choose $nax = 5$ and suppress the output.

\[
\begin{align*}
nax &= 5; X = Flatten[Array[x, {nax + 1, 4 nax + 3}, {0, -2 nax - 1}]] \\
Y &= Select[X, (OddQ[#[[2]]] && Abs[#[[2]]] \leq 2 [[1]] + 1) && Abs[#[[1]]] \leq 2[[1]] + 1) &]
\end{align*}
\]

- $A$ is the amplitude and $p$ the phase of the external periodic force

- The first terms of the perturbation series are:

- The spectral response function is $h[\omega] = \frac{1}{\alpha - \omega^2 + 2i\gamma \omega}$

We insert some numerical values for $\alpha$, $\omega$, $\gamma$, $\lambda$ and calculate the $e^{i\omega t}$-term of the nonlinear response

- The nonlinear Schrödinger equation would contain the term $f(\psi)\psi$ with $f$ given by:

\[
\text{ComplexExpand}[2 \text{Re}[f[A]]]
\]

\[
-2.11765 - 0.75229 A^2 - 0.823776 A^4 - 1.21375 A^6 - 2.0587 A^8 - 3.79189 A^{10}
\]
Plot[Re[f[A]], {A, 0, 1}]
Appendix 5

4.5: Interaction of 2 solitons (analytical)

We use exact results obtained by IST (see section 6) in order to illustrate the interaction of 2 solitons described by an analytical solution of the cubic Schrödinger equation.

\[ n = 2; \quad \kappa = -1; \]
\[ u = \{1, 1\}; \]
\[ v = \{3, -3\}; \]
\[ x_0 = \{-20, 20\}; \]
\[ \phi = \{0, 0\}; \]

\[ g = \text{Table}[\text{Exp}[-x_0[i] + I*\phi[i]], \{i, 1, n\}]; \]
\[ \lambda = \text{Table}[1/2 (I u[i] + v[i]), \{i, 1, n\}]; \]
\[ \gamma := \text{Table}[g[i] \text{Exp}[I \times (\lambda[i] x - \lambda[i]^2 t)], \{i, 1, n\}]; \]
\[ M := \text{Table}[\text{Conjugate}[\frac{1 + \text{Conjugate}[\gamma[j]] \gamma[k]}{\text{Conjugate}[\lambda[j]] - \lambda[k]}], \{j, 1, n\}, \{k, 1, n\}]; \]
\[ \text{zeileM1} = \{\text{Append}[\text{Table}[1, \{i, 1, n\}], 0]\} \]
\[ \text{spalteM1} = \{\gamma\} \]

(* Hänge spalteM1 rechts an M an:*)
\[ \text{Mzwischen} = \text{Transpose}[\{\text{Transpose}[M], \text{spalteM1}\} - \text{Flatten} - 1] \]

(* Hänge zeileM1 unten an Mzwischen an:*)
\[ \text{M1} = \{\text{Mzwischen, zeileM1}\} - \text{Flatten} - 1 \]
\[ \text{M1} // \text{MatrixForm} \]
\[
\text{soliton2}[x_, t_] = \frac{1}{\sqrt{x}} \text{Det}[M1] \\
\left( \left( \frac{3}{10} + \frac{9 i}{10} \right) e^{20 i \cdot (\frac{1}{2} + \frac{3 i}{2} \cdot x)} - \left( \frac{3}{10} - \frac{9 i}{10} \right) e^{20 i \cdot (\frac{1}{2} - \frac{3 i}{2} \cdot x)} \right) + \\
\left( \frac{3}{10} + \frac{9 i}{10} \right) e^i \left( \left( -2 + \frac{3 i}{2} \right) t + \frac{3}{2} - \frac{i}{2} \cdot x \right) - i \text{Conjugate} \left[ \left( -2 + \frac{3 i}{2} \right) t + \left( \frac{3}{2} - \frac{i}{2} \right) \cdot x \right] \right)
\]

- This is the closed form solution without simplifications. Before we can plot it we need to simplify some terms.

\[
S11 = \text{soliton2}[x, t][[1, 1]]; S12 = \text{soliton2}[x, t][[1, 2]];
\]

\[
S13 = \text{soliton2}[x, t][[1, 3]]
\]

\[
s13 = \text{FullSimplify}[\text{soliton2}[x, t][[1, 3]], t \in \text{Reals} \&\& x \in \text{Reals}]
\]

\[
\text{Plot3D}[\text{Abs}[S13/S13], \{x, -50, 50\}, \{t, -30, 0\}, \text{PlotPoints} \rightarrow 100, \text{PlotRange} \rightarrow \{0, 3\}]
\]

\[
S14 = \text{soliton2}[x, t][[1, 4]]
\]

\[
s14 = \text{FullSimplify}[\text{S14, t \in \text{Reals} \&\& x \in \text{Reals}}]
\]

\[
\text{Plot3D}[\text{Abs}[S14/S14], \{x, -50, 50\}, \{t, -30, 0\}, \text{PlotPoints} \rightarrow 100, \text{PlotRange} \rightarrow \{0, 3\}]
\]

\[
S1 = \text{soliton2}[x, t][[1]]
\]

\[
s1 = \text{FullSimplify}[S11 + S12 + s13 + s14, t \in \text{Reals} \&\& x \in \text{Reals}]
\]

\[
\text{Plot3D}[\text{Abs}[s1/S1], \{x, -50, 50\}, \{t, -30, 0\}, \text{PlotPoints} \rightarrow 100, \text{PlotRange} \rightarrow \{0, 3\}]
\]

\[
S2 = \text{soliton2}[x, t][[2, 1]]
\]

\[
S21 = \text{soliton2}[x, t][[2, 1, 1]]
\]

\[
S22 = \text{soliton2}[x, t][[2, 1, 2]]
\]

\[
s22 = \text{FullSimplify}[S22, t \in \text{Reals} \&\& x \in \text{Reals}]
\]

\[
\text{Plot3D}[\text{Abs}[s22/S22], \{x, -50, 50\}, \{t, -30, 0\}, \text{PlotPoints} \rightarrow 100, \text{PlotRange} \rightarrow \{0, 3\}]
\]

\[
S23 = \text{soliton2}[x, t][[2, 1, 3]]
\]

\[
s23 = \text{FullSimplify}[S23, t \in \text{Reals} \&\& x \in \text{Reals}]
\]
Appendix 5

Plot3D[Abs[s23/S23], {x, -50, 50}, {t, -30, 0}, PlotPoints -> 100, PlotRange -> {0, 3}]
S24 = soliton2[x, t][[2, 1, 4]]
s24 = FullSimplify[S24, t \[\in\] Reals && x \[\in\] Reals]
Plot3D[Abs[s24/S24], {x, -50, 50}, {t, -30, 0}, PlotPoints -> 100, PlotRange -> {0, 3}]
S25 = soliton2[x, t][[2, 1, 5]]
s25 = FullSimplify[S25, t \[\in\] Reals && x \[\in\] Reals]
Plot3D[Abs[s25/S25], {x, -50, 50}, {t, -30, 0}, PlotPoints -> 100, PlotRange -> {0, 3}]
S26 = soliton2[x, t][[2, 1, 6]]
s26 = FullSimplify[S26, t \[\in\] Reals && x \[\in\] Reals]
Plot3D[Abs[s26/S26], {x, -50, 50}, {t, -30, 0}, PlotPoints -> 100, PlotRange -> {0, 3}]
S2 = soliton2[x, t][[2, 1]]
s2 = FullSimplify[S21 + s22 + s23 + s24 + s25 + s26, t \[\in\] Reals && x \[\in\] Reals]

TrigToExp[s2]

FullSimplify[\[\frac{1}{10} e^{(-1+3 i) x} + \frac{1}{10} e^{(-1+3 i) x}\]]

\{\{-9/10 - e^{-4/3 t-x} - e^{4/3 t-x} - 9 e^{-2 x}/10 + 1/5 e^x Cos[3 x]\}\} /. \{x \[\rightarrow\] -5, t \[\rightarrow\] -15\} // N,
S2 /. \{x \[\rightarrow\] -5, t \[\rightarrow\] -15\} // N

s2 = -9/10 - e^{-4/3 t-x} - e^{4/3 t-x} - 9 e^{-2 x}/10 + 1/5 e^x Cos[3 x]

Plot3D[Abs[s2/S2], {x, -50, 50}, {t, -30, 0}, PlotPoints -> 100, PlotRange -> {0, 3}]
The following x-t-plot shows two solitons which approach with opposed velocities and fly apart after the interaction without changing their shape.

```
Plot3D[Abs[s1/s2], {x, -50, 50}, {t, -30, 0}, PlotRange -> {0, 1.8}, PlotPoints -> 100]
```
In the density plot of the same solution one discovers the interference pattern of the two solitons in the interaction region as well as the position shift, compared with a situation with two non-interacting solitons.

DensityPlot[Abs[s1/s2], {x, -50, 50}, {t, -30, 0},
PlotPoints -> 200, Mesh -> False, ColorFunctionScaling -> False]